Failure tolerance of synchronization in complex dynamical networks

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A single Hindmarsh-Rose neuron

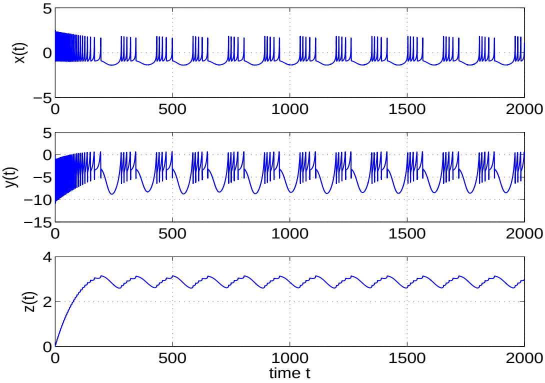
The Hindmarsh-Rose equations for a single neuron:

$$\begin{cases} x'(t) = y - x^3 + 3x^2 - z + \mathrm{I}, \\ y'(t) = 1 - 5x^2 - y, \\ z'(t) = 0.005(4(x + 1.6) - z), \end{cases}$$

where

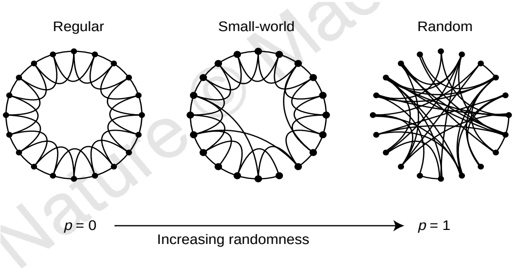
- x is the membrane potential;
- y is associated with the fast current, for example Na^+ or K^+;
- z is associated with the slow current, for example Ca^{2+};
- I is the external current input.

Time evolution of a single HR neuron with $\mathrm{I} = 3.0$



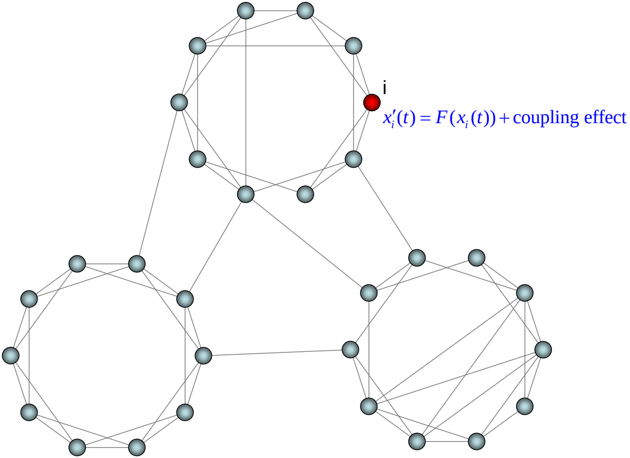
Watts-Strogatz-type small-world networks

- Duncan J. Watts and Steven H. Strogatz, Collective dynamics of 'small-world' networks, *Nature*, 393 (1998), pp. 440-442.



- **Small-world property:** Highly clustered as regular networks and small distance as random networks.

A coupled dynamical network with community structure



Hindmarsh-Rose neural network

- Linearly coupled dynamical network of Hindmarsh-Rose neurons:

$$\mathbf{x}_i'(t) = F(\mathbf{x}_i(t)) - c\sum_{j=1}^{m} \ell_{ij}\Gamma\mathbf{x}_j(t), \quad i = 1, 2, \cdots, m, \tag{1}$$

where

- m is the total number of neurons in the network;
- $\mathbf{x}_i(t) = (x_i(t), y_i(t), z_i(t))^\top$ is the state variable of neuron i;
- $F(\mathbf{x}_i)$ describes the intrinsic dynamics of neuron i, that is,

$$F(\mathbf{x}_i) = \begin{pmatrix} y_i + 3x_i^2 - x_i^3 - z_i + 3.0 \\ 1 - 5x_i^2 - y_i \\ 0.005(4(x_i + 1.6) - z_i) \end{pmatrix};$$

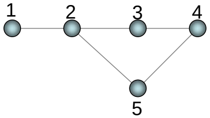
- $c > 0$ is the coupling strength;
- $\Gamma = \mathrm{diag}\{1, 0, 0\}$ is the inner-coupling matrix which determines the coupled components of the neurons.

Laplacian matrix

- Assume that the network is undirected and does not contain self loops.
- The adjacency matrix $A = (a_{ij})_{m\times m}$ of the given network is a real symmetric matrix with

$$\begin{cases} a_{ii} = 0, & \text{for all } i, \\ a_{ij} = a_{ji} = 1, & \text{if the pair of nodes } (i,j) \text{ is connected by a link,} \\ a_{ij} = a_{ji} = 0, & \text{otherwise.} \end{cases}$$

- The matrix $L = (\ell_{ij})_{m\times m}$ is the Laplacian matrix, that is, $L = D - A$, here the matrix $D = \mathrm{diag}\{d_1, d_2, \cdots, d_m\}$ is the degree matrix with $d_i = \sum_{j=1}^{m} a_{ij}$ for $i = 1, 2, \cdots, m$.



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

Properties of the Laplacian matrix

- L is a symmetric matrix with zero row sums and $\ell_{ij} \leq 0$ for all $i \neq j$.
- The zero row sums condition implies that there exists a completely synchronized state

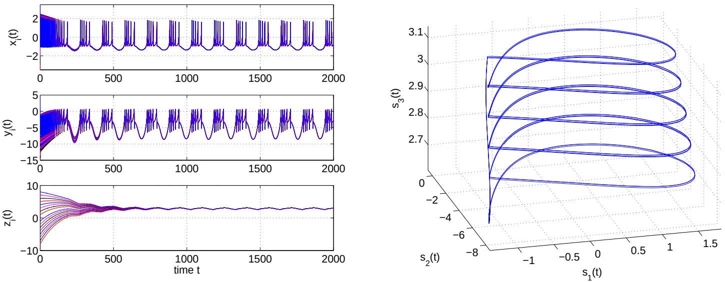
$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_m(t) = \mathbf{s}(t),$$

where $\dot{\mathbf{s}}(t) = F(\mathbf{s}(t))$.

- All the eigenvalues of L are nonnegative;
- L always has at least one zero eigenvalue, say $\lambda_1 = 0$;
- There is only one zero eigenvalue if the network is connected.
- Assume the network is connected $\Longrightarrow 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_m$.

Synchronization analysis based on master stability function method

- Louis M. Pecora and Thomas L. Carroll, Master stability functions for synchronized coupled systems, *Physical Review Letters*, 80 (1998), pp. 2109-2112.
- **Idea:** Consider perturbations around the completely synchronized state. $\Longrightarrow$ We need to know whether the perturbations grow or decay in time.



(left) complete synchronization and (right) synchronized state

Linear stability

- Let $\delta\mathbf{x}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$, for $i = 1, 2, \cdots, m$, be the time evolution of the set of infinitesimal perturbation about $\mathbf{s}(t)$.
- The variational equation of (1) is given by

$$\frac{d}{dt}\delta\mathbf{x}_i = \mathbf{D}F(\mathbf{s})\delta\mathbf{x}_i - c\sum_{j=1}^{m}\ell_{ij}\Gamma\delta\mathbf{x}_i, \quad i = 1, 2, \cdots, m. \tag{2}$$

- Let $\delta\mathbf{y}_i = Q^{-1}\delta\mathbf{x}_i$ for $i = 1, 2, \cdots, m$, we have

$$\frac{d}{dt}\delta\mathbf{y}_i = [\mathbf{D}F(\mathbf{s}) - c\lambda_i\Gamma]\delta\mathbf{y}_i, \quad i = 1, 2, \cdots, m. \tag{3}$$

- For $\lambda_1 = 0$, we have $\frac{d}{dt}\delta\mathbf{y}_1 = \mathbf{D}F(\mathbf{s})\delta\mathbf{y}_1$ which corresponds the perturbation parallel to the synchronized state.
- The other $m-1$ systems correspond to transverse directions and should be damped out to have a sychronized state.

Master stability function

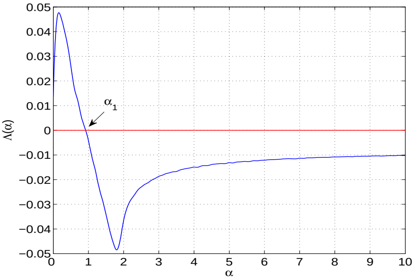
- **Observation:** Each system of (3) has the same form with only the parameter $\alpha_i = c\lambda_i$ being different.
- This leads to the master stability equation

$$\mathbf{y}' = [\mathbf{D}F(\mathbf{s}) - \alpha\Gamma]\mathbf{y}. \tag{4}$$

- Calculate the largest Lyapunov exponent $\Lambda(\alpha)$, which is called the master stability function (MSF).
- For large t, $\|\mathbf{y}(t)\| \approx \|\mathbf{y}(0)\| \exp(\Lambda(\alpha)t)$ and $\mathbf{y} \to 0$ if $\Lambda(\alpha) < 0$.
- Note that the MSF being negative is a necessary but not sufficient condition for synchronization to actually occur.

The MSF for Hindmarsh-Rose neuron

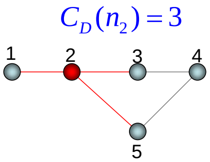
- The MSF for Hindmarsh-Rose neuron:



- $\Lambda(\alpha)$ is negative on the interval (α_1, ∞), where $\alpha_1 \approx 0.9250$.
- The condition for synchronization becomes $c\lambda_2 > \alpha_1$.
- λ_2 can be used as an indicator of the synchronizability of the network.
- **Question:** How the network structures affect the synchronizability?

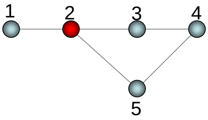
Centrality measure – degree centrality

- **Centrality:** Which are the most important or central node in a network?
- **Degree centrality:** $C_D(n_i) = \sum_{j=1}^{m} a_{ij}$.
- An important node is involved in large number of interactions.



Centrality measure – eigenvector centrality

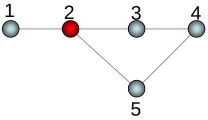
- **Eigenvector centrality:**
  - Let μ be the largest eigenvalue of A with corresponding eigenvector $V = (v_1, v_2, \cdots, v_m)^\top$, that is, $AV = \mu V$.
  - V is the eigenvector centrality.
- Specifically, $C_E(n_i) = v_i = \mu^{-1} \sum_{j=1}^{m} a_{ij} v_j$.
- Perron-Frobenius theorem $\Longrightarrow \mu > 0$ and $v_i > 0, \forall i$
- An important node is connected to important neighbors.



$$V = \begin{pmatrix} 0.2610 \\ 0.5573 \\ 0.4647 \\ 0.4352 \\ 0.4674 \end{pmatrix} \begin{array}{l} \leftarrow C_E(n_1) \\ \leftarrow C_E(n_2) \\ \leftarrow C_E(n_3) \\ \leftarrow C_E(n_4) \\ \leftarrow C_E(n_5) \end{array}$$

Centrality measure – closeness centrality

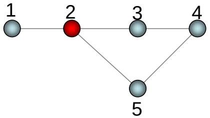
- **Closeness centrality:**
  - p_{ij} = the length of the shortest path from node i to node j.
  - The mean distance from node i to other nodes is $\overline{p}_i = \frac{1}{m}\sum_{j=1}^{m} p_{ij}$.
  - $C_C(n_i) = \frac{1}{\overline{p}_i}$.
- An important node is typically "close" to, and can communicate quickly with, the other nodes in the network.



$$\begin{aligned}\overline{p}_2 &= \tfrac{1}{5}(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) \\ &= \tfrac{1}{5}(1+0+1+2+1) \\ &= 1\end{aligned}$$

Centrality measure – betweenness centrality

- **Betweenness centrality:**
  - σ_{uv} = total number of shortest paths from node u to node v.
  - $\sigma_{uv}(n_i)$ = the number of shortest paths from node u to node v that pass through node i.
  - $C_B(n_i) = \sum_{u \neq n_i \neq v} \frac{\sigma_{uv}(n_i)}{\sigma_{uv}}$.
- An important node will lie on a high proportion of paths between other nodes in the network.



$n_1 \leftrightarrow n_2 : n_1n_2 \Rightarrow \frac{0}{1}$

$n_1 \leftrightarrow n_3 : n_1n_2n_3 \Rightarrow \frac{1}{1}$

$n_1 \leftrightarrow n_4 : n_1n_2n_3n_4; n_1n_2n_5n_4 \Rightarrow \frac{2}{2}$

$n_1 \leftrightarrow n_5 : n_1n_2n_5 \Rightarrow \frac{1}{1}$

$n_2 \leftrightarrow n_3 : n_2n_3 \Rightarrow \frac{0}{1}$

$n_2 \leftrightarrow n_4 : n_2n_3n_4; n_2n_5n_4 \Rightarrow \frac{0}{2}$

$n_2 \leftrightarrow n_5 : n_2n_5 \Rightarrow \frac{0}{1}$

$n_3 \leftrightarrow n_4 : n_3n_4 \Rightarrow \frac{0}{1}$

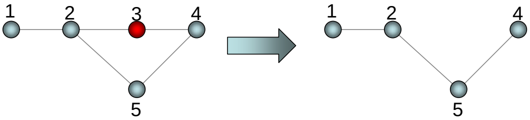
$n_3 \leftrightarrow n_5 : n_3n_4n_5; n_3n_2n_5 \Rightarrow \frac{1}{2}$

$n_4 \leftrightarrow n_5 : n_4n_5 \Rightarrow \frac{0}{1}$

$$\therefore C_B(n_2) = 2\left(\frac{0}{1}+\frac{1}{1}+\frac{2}{2}+\frac{1}{1}+\frac{0}{1}+\frac{0}{2}+\frac{0}{1}+\frac{0}{1}+\frac{1}{2}+\frac{0}{1}\right) = 7$$

Component failure

- Netwroks may undergo failures in one or more of their components, i.e., nodes and/or edges.
- The failures are of two types in general:
  - Random failure $\Leftrightarrow$ errors, or
  - Systematic failure $\Leftrightarrow$ attacks.
- **Objective:** We will discuss how random and systematic failures in the nodes of a network influence its synchronizability.

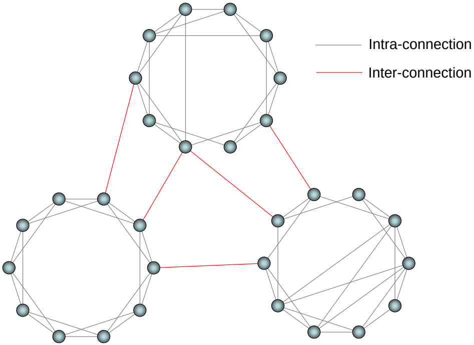


Strategies

- To choose candidate nodes for removal, five strategies were considered:
  1. Random failure
  2. Systematic failure based on degree centrality
  3. Systematic failure based on eigenvector centrality
  4. Systematic failure based on closeness centrality
  5. Systematic failure based on betweenness centrality

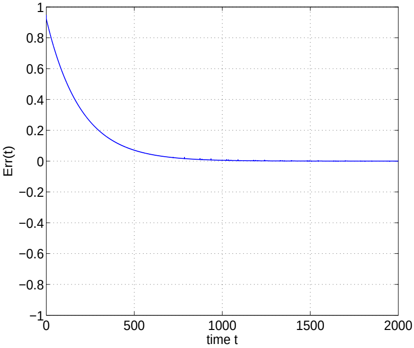
Connection networks

- We considered artificially constructed model networks:
  - Total number of nodes $m = 400$.
  - The number of clusters is $M = 20$ and each of which is a WS-type network with intra-connection probability P_{intra}.
  - The M clusters are arranged on a ring and the inter-connections between different clusters exist randomly with the probability P_{inter}.



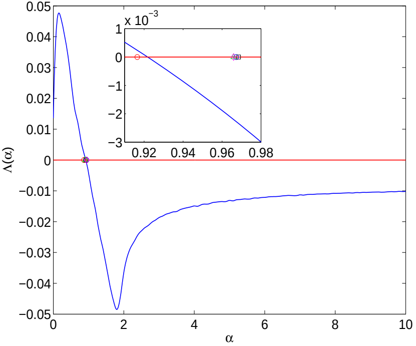
An example

- Here, we choose a network with eigenvalue $\lambda_2(A) = 0.0727$.
- Let $c = 13 \Longrightarrow c\lambda_2(A) \in (\alpha_1, \infty)$.
- The waveforms of $\mathrm{Err}(t) = \max_{i<j} \|x_i(t) - x_j(t)\|_\infty$ is depicted as follows:

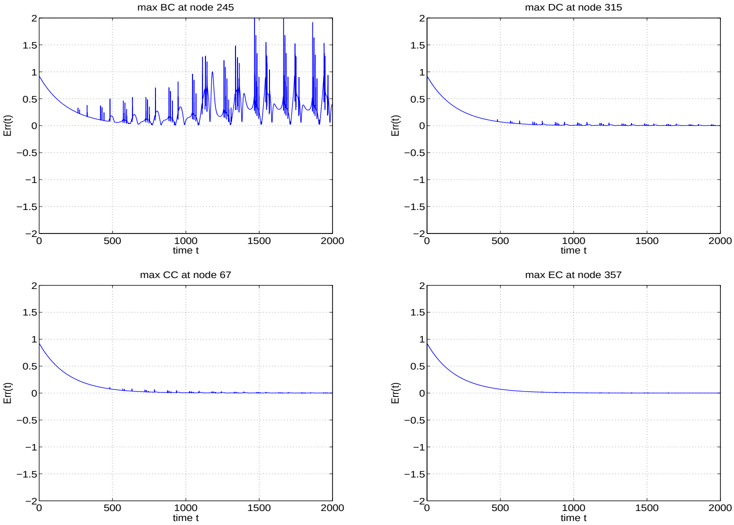


An example (continued)

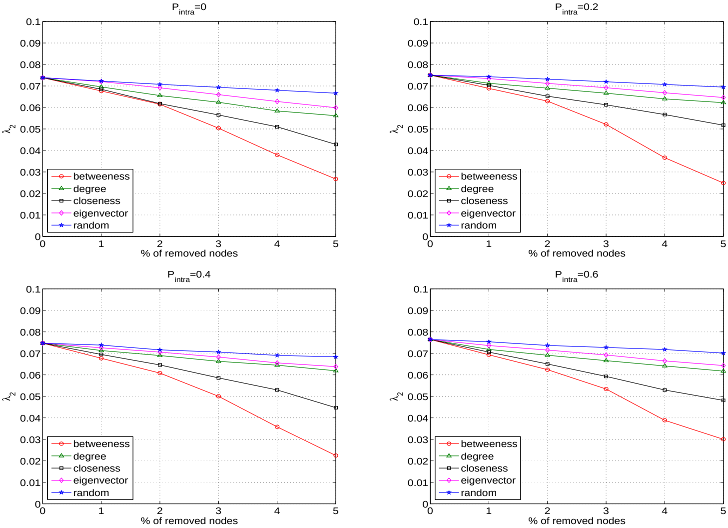
- Based on different systematic failure strategies, we removed one node and observed the location of $c\lambda_2$.



An example (continued)

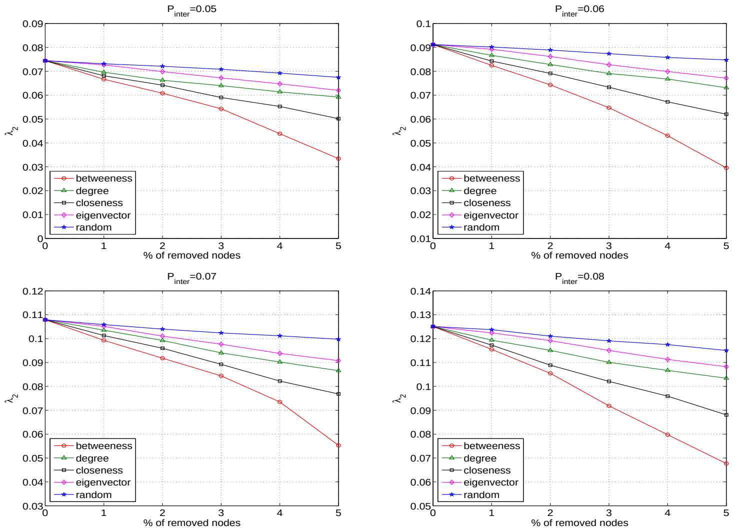


Inter-connection probability $P_{inter} = 0.05$



All data points are averaged over 20 network realizations

Intra-connection probability $P_{\text{intra}} = 0.1$



All data points are averaged over 20 network realizations

Concluding remarks

- We have studied how random and systematic failures in the nodes of a network influence its synchronizability.
- Failures based on the betweenness centrality, that is, removing the nodes with high values of the betweenness centrality, has the significant effect on the network synchronizability.
- The synchronizability of the constructed networks is robust against random removal of nodes.

Thank you for your attention!

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