An overview of projection methods for viscous incompressible flow (I)

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Governing equations

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) and $[0, T]$ be the time interval. The time-dependent, incompressible Stokes problem can be posed as: find $\boldsymbol{u}$ and p with $\int_\Omega p dV = 0$, so that

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \times (0, T], \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0, T], \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega \times [0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t = 0\}. \end{aligned}$$

- $\boldsymbol{u}$ is the velocity field, p the pressure (divided by a constant density ρ), ν the kinematic viscosity, and $\boldsymbol{f}$ the body force.

Time-discretization of the incompressible Stokes eqns

First, we discretize the time variable of the Stokes problem, with the spatial variable being left continuous. Consider the implicit Euler time-discretization:

$$\begin{aligned}
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{0} \quad \text{on } \partial\Omega,
\end{aligned}$$

where $t_i := i\Delta t$ for $i = 0, 1, \cdots, K = \frac{T}{\Delta t}, \cdots$, $\Delta t > 0$ is the time step length, and $\boldsymbol{g}^n$ denotes an approximate (or exact) value of $\boldsymbol{g}(t_n)$ at the time level n.

It is highly inefficient in solving this coupled system of Stokes equations directly. This is precisely the reason for proposing the projection approach to decouple the computation of $(\boldsymbol{u}^{n+1}, p^{n+1})$.

Idea of projection method

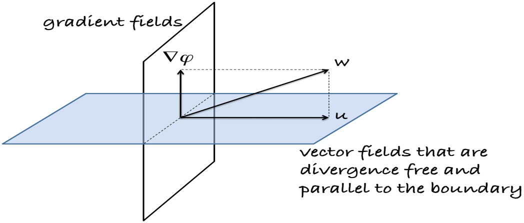
1. This approach solves the equations of time-discretization of the incompressible (Navier-)Stokes equations.
2. The underlying idea of projection method, first introduced by Chorin (1968, 1969) and Temam (1969), is based on applying the Helmholtz-Hodge decomposition (HHD) to the time-discretized incompressible Navier-Stokes equations.
3. The feature of projection method is to compute velocity and pressure fields separately through the computation of an intermediate velocity $\widetilde{u}$, and then project it onto the space of divergence-free vector fields.

Helmholtz-Hodge decomposition (Chorin & Marsden)

A smooth vector field $\boldsymbol{w}$ defined on $\overline{\Omega}$ can be uniquely decomposed orthogonally in the form:

$$\boldsymbol{w} = \boldsymbol{u} + \nabla\varphi,$$

where $\boldsymbol{u}$ has zero divergence, $\nabla \cdot \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.



Remarks:

- Orthogonality means $\int_\Omega \boldsymbol{u} \cdot \nabla\varphi dV = 0$ (L^2-inner product).
- The HHD describes the decomposition of a flow field $\boldsymbol{w}$ into its divergence-free component $\boldsymbol{u}$ and curl-free component $\nabla\varphi$, since $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \times (\nabla\varphi) = 0$ in Ω.

Proof of HHD Theorem

Existence: Given a smooth vector field $\boldsymbol{w}$, let φ be defined as the solution to the Neumann problem

$$\begin{cases} \nabla^2\varphi = \nabla\cdot\boldsymbol{w} & \text{in } \Omega, \\ \nabla\varphi\cdot\boldsymbol{n} = \boldsymbol{w}\cdot\boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

It is known that the solution φ of this problem exists and is defined up to an arbitrary additive constant, see the Remark below. Define $\boldsymbol{u} := \boldsymbol{w} - \nabla\varphi$, then it is obvious that $\nabla\cdot\boldsymbol{u} = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$.

Remark: Consider the Neumann problem on a smooth domain D,

$$\begin{cases} \nabla^2\psi = f & \text{in } D, \\ \nabla\psi\cdot\boldsymbol{n} = g & \text{on } \partial D. \end{cases}$$

The problem has a unique solution up to a constant if and only if the following compatibility condition holds:

$$\int_D f dV = \int_D \nabla\cdot\nabla\psi dV = \int_{\partial D} \nabla\psi\cdot\boldsymbol{n} dA = \int_{\partial D} g dA.$$

Proof of HHD Theorem (cont.)

Orthogonality of $\boldsymbol{u}$ and $\nabla\varphi$: First, note that

$$\nabla \cdot (\varphi \boldsymbol{u}) = (\nabla \cdot \boldsymbol{u})\varphi + \boldsymbol{u} \cdot \nabla\varphi.$$

Then by $\nabla \cdot \boldsymbol{u} = 0$ in Ω, Divergence Theorem, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$,

$$\int_\Omega \boldsymbol{u} \cdot \nabla\varphi dV = \int_\Omega \nabla \cdot (\varphi \boldsymbol{u}) dV = \int_{\partial\Omega} (\varphi \boldsymbol{u}) \cdot \boldsymbol{n} dA = \int_{\partial\Omega} \varphi(\boldsymbol{u} \cdot \boldsymbol{n}) dA = 0. \quad (\clubsuit)$$

Uniqueness: Suppose $\boldsymbol{w} = \boldsymbol{u}_i + \nabla\varphi_i$, $\nabla \cdot \boldsymbol{u}_i = 0$ in Ω and $\boldsymbol{u}_i \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ for $i = 1, 2$. Then

$$(\boldsymbol{u}_1 - \boldsymbol{u}_2) + \nabla(\varphi_1 - \varphi_2) = 0 \quad \text{in } \Omega.$$

Taking the inner product with $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have

$$\begin{aligned} 0 &= \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) + (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \nabla(\varphi_1 - \varphi_2) dV \\ &= \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) + 0\, dV. \quad (\text{using } (\clubsuit) \text{ again}) \end{aligned}$$

It follows that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\nabla\varphi_1 = \nabla\varphi_2$.

Non-incremental pressure-correction scheme

The simplest pressure-correction scheme has originally been proposed by Chorin (1968 & 1969). The algorithm is as follows:

Step 1: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\Delta t} + \nabla p^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Non-incremental pressure-correction scheme

- Step 2 is equivalent to solving the following pressure-Poisson equation with the homogeneous Neumann boundary condition:

$$\begin{cases} \nabla^2 p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \widetilde{\boldsymbol{u}}^{n+1} & \text{in } \Omega, \\ \nabla p^{n+1} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

  and then define the velocity field by $\boldsymbol{u}^{n+1} = \widetilde{\boldsymbol{u}}^{n+1} - \Delta t \nabla p^{n+1}$.

- The second step is usually referred to as the projection step.

$$\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{u}^{n+1} + \Delta t \nabla p^{n+1} = \boldsymbol{u}^{n+1} + \nabla(\Delta t p^{n+1}).$$

  This is indeed the standard HHD of $\widetilde{\boldsymbol{u}}^{n+1}$.

Non-incremental pressure-correction scheme

- We observe that the boundary condition $\nabla p^{n+1} \cdot \boldsymbol{n} = 0$ in step 2 is enforce on the pressure. Rannacher (1991) showed that this artificial Neumann boundary condition induces a numerical boundary layer on the pressure.

Theorem (Prohl (1997), Rannacher (1991), Shen (1992))

Assuming that $(\boldsymbol{u}^e, p^e)$, solving the Stokes equations, is sufficiently smooth, the solution of above projection method, satisfies the following error estimates:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} \leq c(\boldsymbol{u}^e, p^e, T)\Delta t,$$

$$\|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} \leq c(\boldsymbol{u}^e, p^e, T)\Delta t^{1/2}.$$

Note: We denote that $\varphi_{\Delta t} = \{\varphi^0, \varphi^1, \cdots, \varphi^K\}$ be some sequence of functions in a Hilbert space E and define the following discrete norm:

$$\|\varphi_{\Delta t}\|_{\ell^2(E)} = \left(\Delta t \sum_{k=1}^{K} \|\varphi^k\|_E^2\right)^{1/2}, \quad \|\varphi_{\Delta t}\|_{\ell^\infty(E)} = \max_{0\leq k\leq K} \|\varphi^k\|_E.$$

Standard incremental pressure-correction schemes

- Goda (1979) observed (probably first) that the pressure gradient is obviously missing in first step. He introduced an old value of the pressure gradient ∇p^n in the first step:

  Step 1: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{n+1} + \nabla p^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

  Step 2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\Delta t} + \nabla p^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

- This idea was made popular by Van Kan (1986) who proposed a second-order incremental pressure-correction scheme.

Standard incremental pressure-correction schemes

Using the BDF of second-order to approximate the time derivative, the incremental pressure-correction scheme reads as follows:

Step 1: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$\begin{cases} \dfrac{3\widetilde{\boldsymbol{u}}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \nu\nabla^2\widetilde{\boldsymbol{u}}^{n+1} + \nabla p^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{3\boldsymbol{u}^{n+1} - 3\widetilde{\boldsymbol{u}}^{n+1}}{2\Delta t} + \nabla(p^{n+1} - p^n) = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Standard incremental pressure-correction schemes

The scheme needs to be initialized with $(\widetilde{\boldsymbol{u}}^1, \boldsymbol{u}^1, p^1)$ and we make the following hypothesis:

$(\widetilde{\boldsymbol{u}}^1, \boldsymbol{u}^1, p^1)$ is computed such that the following estimates holds:

$$
\begin{aligned}
\|\boldsymbol{u}^e(\Delta t) - \widetilde{\boldsymbol{u}}^1\|_0 &\le c\Delta t^2 \\
\|\boldsymbol{u}^e(\Delta t) - \widetilde{\boldsymbol{u}}^1\|_1 &\le c\Delta t^{3/2} \\
\|p^e(\Delta t) - p^1\|_0 &\le c\Delta t
\end{aligned}
$$

Theorem (Shen (1996), E and Liu (1995), Guermond (1999))

Under the hypothesis , if the $(\boldsymbol{u}^e, p^e)$, solving the Stokes equations, is smooth enough in space and time, the solution of above projection method satisfies the following error estimates:

$$
\begin{aligned}
\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} &\le c(\boldsymbol{u}^e, p^e, T)\Delta t^2, \\
\|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} &\le c(\boldsymbol{u}^e, p^e, T)\Delta t.
\end{aligned}
$$

Rotational incremental pressure-correction schemes

- It can observe that $\nabla(p^{n+1} - p^n) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ which implies that

$$\nabla p^{n+1} \cdot \boldsymbol{n} = \nabla p^n \cdot \boldsymbol{n} = \cdots = \nabla p^0 \cdot \boldsymbol{n} \quad \text{on } \partial\Omega.$$

  It is non-physical Neumann boundary condition enforced on the pressure that introduces the numerical boundary layer referred to above and consequently limits the accuracy of the scheme.

- To overcome the difficulty caused by the artificial pressure Neumann boundary condition, Timmermans, Minev and Van De Vosse (1996) slightly modify this algorithm which is referred as the incremental pressure-correction scheme in rotational form.

Rotational incremental pressure-correction schemes

Step 1: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$\begin{cases} \dfrac{3\widetilde{\boldsymbol{u}}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{n+1} + \nabla p^n & = \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\ \widetilde{\boldsymbol{u}}^{n+1} & = \boldsymbol{0} \quad \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and ϕ^{n+1} by solving

$$\begin{cases} \dfrac{3\boldsymbol{u}^{n+1} - 3\widetilde{\boldsymbol{u}}^{n+1}}{2\Delta t} + \nabla \phi^{n+1} & = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} & = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} & = \boldsymbol{0} \quad \text{on } \partial\Omega. \end{cases}$$

Step 3: Update p^{n+1} by

$$p^{n+1} = p^n + \phi^{n+1} - \nu \nabla \cdot \widetilde{\boldsymbol{u}}^{n+1}.$$

Rotational incremental pressure-correction schemes

- Summing all equations of above step 1 to 3, we have

$$
\begin{aligned}
\frac{3\boldsymbol{u}^{n+1} - 4\boldsymbol{u}^{n} + \boldsymbol{u}^{n-1}}{2\Delta t} + \nu \nabla \times \nabla \times \boldsymbol{u}^{n+1} + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \partial\Omega,
\end{aligned}
$$

by the facts that $\nabla^2 \widetilde{\boldsymbol{u}}^{n+1} = \nabla(\nabla \cdot \widetilde{\boldsymbol{u}}^{n+1}) - \nabla \times \nabla \times \widetilde{\boldsymbol{u}}^{n+1}$ and $\nabla \times \nabla \times \widetilde{\boldsymbol{u}}^{n+1} = \nabla \times \nabla \times \boldsymbol{u}^{n+1}$.

- We observe that

$$
\nabla p^{n+1} \cdot \boldsymbol{n} = (\boldsymbol{f}^{n+1} - \nu \nabla \times \nabla \times \boldsymbol{u}^{n+1}) \cdot \boldsymbol{n}
$$

is a consistent pressure boundary condition.

Rotational incremental pressure-correction schemes

Theorem (Guermond and Shen (2004))

Assume that the initialization hypothesis holds. Provided the $(\boldsymbol{u}^e, p^e)$, solving the Stokes equations, is smooth enough in space and time, the solution of above method satisfies the estimates

$$\begin{aligned}
&\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} \le c(\boldsymbol{u}^e, p^e, T)\Delta t^2, \\
&\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^2([H^1(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^2([H^1(\Omega)]^d)} + \|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^2(L^2(\Omega))} \\
&\qquad \le c(\boldsymbol{u}^e, p^e, T)\Delta t^{3/2}.
\end{aligned}$$

Relation with other schemes

Kim and Moin (1985) propose to apply $\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{u}^{n+1} + \frac{2\Delta t}{3}\nabla\phi^n$ to the boundary conditions of $\widetilde{\boldsymbol{u}}^{n+1}$, in order to obtain second-order accuracy in the velocity. The scheme can be written as follows:

Step 1: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$\begin{cases} \dfrac{3\widetilde{\boldsymbol{u}}^{n+1} - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \nu\nabla^2\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{n} = 0 \ \ \& \ \ \widetilde{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\tau} = \dfrac{2\Delta t}{3}\nabla\phi^n\cdot\boldsymbol{\tau} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{3\boldsymbol{u}^{n+1} - 3\widetilde{\boldsymbol{u}}^{n+1}}{2\Delta t} + \nabla\phi^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Step 3: Update the pressure, $p^{n+1} = \phi^{n+1} - \nu\nabla\cdot\widetilde{\boldsymbol{u}}^{n+1}$

Kim-Moin and rotational form of the pressure-correction methods are "equivalent!"

Staring with the Kim-Moin method and changing the variables,

$$
\begin{aligned}
\boldsymbol{u}^* &= \widetilde{\boldsymbol{u}} - \frac{2\Delta t}{3}\nabla\phi^n \cdot \boldsymbol{\tau}, \qquad \psi^{n+1} = \phi^{n+1} - \phi^n, \\
p^{n+1} &= \phi^{n+1} - \frac{3\nu\Delta t}{2}\nabla^2\varphi^{n+1}, \qquad p^{n-1} = \phi^{n-1} - \frac{3\nu\Delta t}{2}\nabla^2\phi^{n-1},
\end{aligned}
$$

we can find that solution $(\boldsymbol{u}^{n+1}, p^{n+1})$ of the Kim-Moin method also solves the rotational form of the pressure-correction methods and *vice versa*:

$$
\left\{
\begin{aligned}
\frac{3\boldsymbol{u}^* - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} - \nu\nabla^2\boldsymbol{u}^* + \nabla p^n &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\boldsymbol{u}^* &= \mathbf{0} \quad \text{on } \partial\Omega.
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\frac{3\boldsymbol{u}^{n+1} - 3\boldsymbol{u}^*}{2\Delta t} + \nabla\psi^{n+1} &= \mathbf{0} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} &= 0 \quad \text{on } \partial\Omega;
\end{aligned}
\right.
$$

$$
p^{n+1} = p^{n-1} + \psi^{n+1} - \frac{3\nu\Delta t}{2}\nabla^2\psi^{n+1}.
$$

Numerical tests

Example 1: A comparison between the standard and the rotational forms of the projection methods.

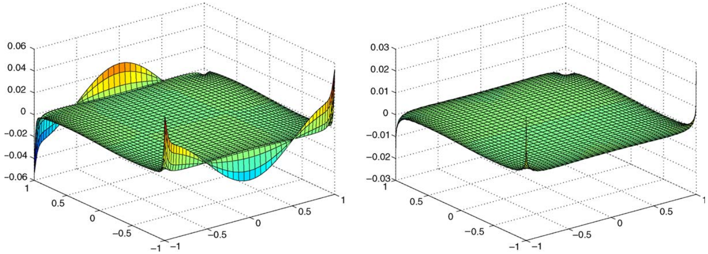


Fig. 1. Pressure error field at time $t = 1$ in a square: (left) standard form; (right) rotational form.

This test suggests that the rotational form successfully improves the numerical boundary layer issue.

Numerical tests

Example 2: A comparison between the standard and the rotational forms of the projection methods in a periodic channel. The channel is periodic in the x direction.

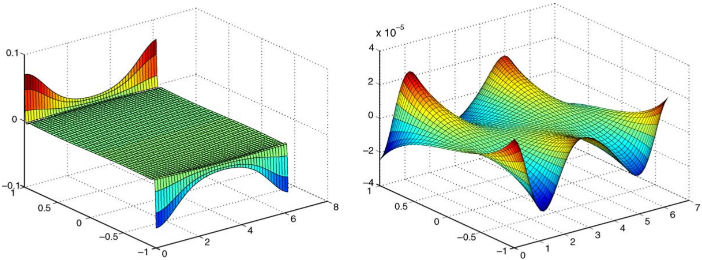


Fig. 2. Error field on pressure at time $t = 1$ in a channel: (left) standard form; (right) rotational form.

It maybe can be conjectured that the large errors occurring at the corners of the square domain are due to the lack of smoothness of the domain.

Numerical tests

Example 3: Convergence tests using $P_2/P1$ finite element for rotational forms of the projection methods.

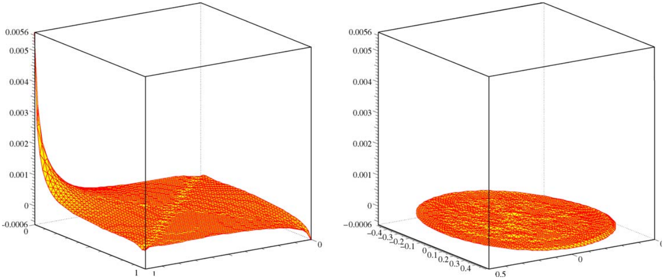


Fig. 3. Error field on pressure in a rectangular domain (left) and on a circular domain (right).

The pressure field on the circular domain is free of numerical boundary layer, whereas large errors are still present at the corners of the domain.

Numerical tests

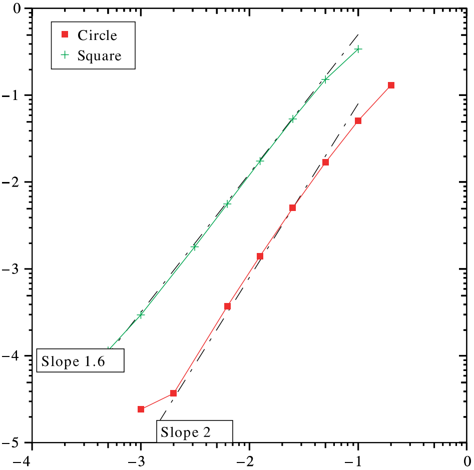


Fig. 4. Comparison of convergence rates on pressure in L^∞-norm at $T = 2$: (■) for the circular domain; (+) for the square.

Thank you for your attention!