An overview of projection methods for viscous incompressible flow (II)

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Governing equations

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) and $[0, T]$ be the time interval. The time-dependent, incompressible Stokes problem can be posed as: find $\boldsymbol{u}$ and p with $\int_\Omega p dV = 0$, so that

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \times (0, T], \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0, T], \\
\boldsymbol{u} &= \mathbf{0} \quad \text{on } \partial\Omega \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t = 0\}.
\end{aligned}
$$

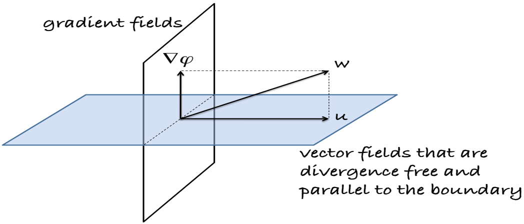
- $\boldsymbol{u}$ is the velocity field, p the pressure (divided by a constant density ρ), ν the kinematic viscosity, and $\boldsymbol{f}$ the body force.

Helmholtz-Hodge decomposition (Chorin & Marsden)

A smooth vector field $\boldsymbol{w}$ defined on $\overline{\Omega}$ can be uniquely decomposed orthogonally in the form:

$$\boldsymbol{w} = \boldsymbol{u} + \nabla \varphi,$$

where $\boldsymbol{u}$ has zero divergence, $\nabla \cdot \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.



Remarks:

- Orthogonality means $\int_\Omega \boldsymbol{u} \cdot \nabla \varphi dV = 0$ (L^2-inner product).
- The HHD describes the decomposition of a flow field $\boldsymbol{w}$ into its divergence-free component $\boldsymbol{u}$ and curl-free component $\nabla \varphi$, since $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \times (\nabla \varphi) = 0$ in Ω.

Time-discretization of the incompressible Stokes eqns

First, we discretize the time variable of the Stokes problem, with the spatial variable being left continuous. Consider the qth-order backward difference formula (BDFq):

$$\begin{aligned} \frac{1}{\Delta t}\left(\beta_q \boldsymbol{u}^{n+1} - \sum_{j=0}^{q-1} \beta_j \boldsymbol{u}^{n-j}\right) - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned}$$

where $t_i := i\Delta t$ for $i = 0, 1, \cdots, K = \frac{T}{\Delta t}, \cdots$, $\Delta t > 0$ is the time step length, and $\boldsymbol{g}^n$ denotes an approximate (or exact) value of $\boldsymbol{g}(t_n)$ at the time level n.

A general pressure-correction projection methods

Step 1: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}$,

$$\begin{cases} \dfrac{1}{\Delta t}\left(\beta_q \widetilde{\boldsymbol{u}}^{n+1} - \sum_{j=0}^{q-1} \beta_j \boldsymbol{u}^{n-j}\right) - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{n+1} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases}$$

where $p^{\star,n+1} = \sum_{j=0}^{r-1} \gamma_j p^{n-j}$ is the rth order extrapolation of p^{n+1}.

Step 2: Determine $\boldsymbol{u}^{n+1}$ and ϕ^{n+1} by solving

$$\begin{cases} \dfrac{\beta_q}{\Delta t}(\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}) + \nabla \phi^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Step 3: Update the pressure, $p^{n+1} = p^{\star,n+1} + \phi^{n+1} - \chi \nu \nabla \cdot \widetilde{\boldsymbol{u}}^{n+1}$, χ is a user-defined coefficient that may be equal to 0 or 1.

A general pressure-correction projection methods

- $(q, r) = (1, 0)$ and $\chi = 0$

  $\Rightarrow$ non-incremental pressure-correction scheme (Chorin).
- $(q, r) = (2, 1)$ and $\chi = 0$

  $\Rightarrow$ standard pressure-correction scheme.
- $(q, r) = (2, 1)$ and $\chi = 1$

  $\Rightarrow$ rotational pressure-correction scheme. ☺

Remark: If one chooses $r = q$, then the formal consistency errors for the velocity in H^1-norm and the pressure in L^2-norm are both of the same order. However, stability and convergence are only available for $q = r = 1$.

Velocity-correction schemes

- These schemes have been introduced in a somewhat different (although equivalent) form by Orszag et al. (1986) and Karniadakis et al. (1991).
- The main idea is to switch the role of the velocity and the pressure in the pressure-correction schemes, i.e., the viscous term is treated explicitly or ignored in the first substep and the velocity is corrected accordingly in the second substep.

Non-incremental velocity-correction scheme

Set $\widetilde{\boldsymbol{u}}^0 = \boldsymbol{u}_0$, and for $n \geq 0$ find $(\widetilde{\boldsymbol{u}}^{n+1}, \boldsymbol{u}^{n+1}, p^{n+1})$ by the following algorithm:

Step 1: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{1}{\Delta t}(\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^n) + \nabla p^{n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$\begin{cases} \dfrac{1}{\Delta t}(\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^{n+1}) - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Non-incremental velocity-correction scheme

Theorem (Rannacher (1991), Guermond & Shen (2003))

Assuming that $(\boldsymbol{u}^e, p^e)$, solving the Stokes equations, is sufficiently smooth, the solution of above projection method, satisfies the following error estimates:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} \leq c(\boldsymbol{u}^e, p^e, T)\Delta t,$$

$$\|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} \leq c(\boldsymbol{u}^e, p^e, T)\Delta t^{1/2}.$$

Note: We denote that $\varphi_{\Delta t} = \{\varphi^0, \varphi^1, \cdots, \varphi^K\}$ be some sequence of functions in a Hilbert space E and define the following discrete norm:

$$\|\varphi_{\Delta t}\|_{\ell^2(E)} = \left(\Delta t \sum_{k=1}^{K} \|\varphi^k\|_E^2\right)^{1/2}, \quad \|\varphi_{\Delta t}\|_{\ell^\infty(E)} = \max_{0\leq k\leq K} \|\varphi^k\|_E.$$

Standard velocity-correction scheme

Using the BDFq to approximate the time derivative, the standard velocity-correction scheme reads as follows:

Step 1: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$
\left\{
\begin{array}{rcl}
\frac{1}{\Delta t}\left(\beta_q \boldsymbol{u}^{n+1} - \sum_{j=0}^{q-1} \beta_j \widetilde{\boldsymbol{u}}^{n-j}\right) - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{\star,n+1} + \nabla p^{n+1} & = & \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} & = & 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} & = & \boldsymbol{0} \quad \text{on } \partial\Omega.
\end{array}
\right.
$$

where $\widetilde{\boldsymbol{u}}^{\star,n+1} = \sum_{j=0}^{r-1} \gamma_j \widetilde{\boldsymbol{u}}^{n-j}$ be a rth order extrapolation of $\widetilde{\boldsymbol{u}}^{n+1}$.

Step 2: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$
\left\{
\begin{array}{rcl}
\frac{\beta_q}{\Delta t}(\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^{n+1}) - \nu \nabla^2 (\widetilde{\boldsymbol{u}}^{n+1} - \widetilde{\boldsymbol{u}}^{\star,n+1}) & = & \boldsymbol{0} \quad \text{in } \Omega, \\
\widetilde{\boldsymbol{u}}^{n+1} & = & \boldsymbol{0} \quad \text{on } \partial\Omega.
\end{array}
\right.
$$

Standard velocity-correction scheme

Hypothesis

$\widetilde{\boldsymbol{u}}^1$ is computed such that the following estimates holds:

$$\|\boldsymbol{u}^e(\Delta t) - \widetilde{\boldsymbol{u}}^1\|_0 \le c\Delta t^2,$$
$$\|\boldsymbol{u}^e(\Delta t) - \widetilde{\boldsymbol{u}}^1\|_1 \le c\Delta t^{3/2},$$
$$\|\boldsymbol{u}^e(\Delta t) - \widetilde{\boldsymbol{u}}^1\|_2 \le c\Delta t.$$

Theorem (Guermond & Shen (2003))

Under the hypothesis , if the $(\boldsymbol{u}^e, p^e)$, solving the Stokes equations, is smooth enough in space and time, the solution of above projection method with $(q, r) = (2, 1)$ satisfies the following error estimates:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} \le c(\boldsymbol{u}^e, p^e, T)\Delta t^2,$$
$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([L^\in(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} \le c(\boldsymbol{u}^e, p^e, T)\Delta t^{\frac{3}{2}},$$
$$\|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} + \|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} \le c(\boldsymbol{u}^e, p^e, T)\Delta t.$$

Rotational form of the velocity-correction scheme

In order to obtain a better approximation of the pressure, Guermond and Shen (2001, 2003) propose to replace $\nabla^2 \widetilde{\boldsymbol{u}}^{\star,n+1}$ by $-\nabla \times \nabla \times \widetilde{\boldsymbol{u}}^{\star,n+1}$:

Step 1: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$
\begin{cases}
\dfrac{1}{\Delta t}\Big(\beta_q \boldsymbol{u}^{n+1} - \displaystyle\sum_{j=0}^{q-1} \beta_j \widetilde{\boldsymbol{u}}^{n-j}\Big) + \nu \nabla \times \nabla \times \widetilde{\boldsymbol{u}}^{\star,n+1} + \nabla p^{n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\
\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\Omega.
\end{cases}
$$

Step 2: Solve for the intermediate velocity field $\widetilde{\boldsymbol{u}}^{n+1}$,

$$
\begin{cases}
\dfrac{\beta_q}{\Delta t}(\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^{n+1}) - \nu \nabla^2 \widetilde{\boldsymbol{u}}^{n+1} - \nu \nabla \times \nabla \times \widetilde{\boldsymbol{u}}^{\star,n+1} = \boldsymbol{0} & \text{in } \Omega, \\
\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{0} & \text{on } \partial\Omega.
\end{cases}
$$

This scheme is referred to as the rotational form of the velocity-correction algorithm.

Rotational form of the velocity-correction scheme

Theorem (Guermond & Shen (2003))

Under the hypothesis , if the $(\boldsymbol{u}^e, p^e)$*, solving the Stokes equations, is smooth enough in space and time, the solution of above projection method with* $(q, r) = (2, 1)$ *satisfies the following error estimates:*

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} \leq c(\boldsymbol{u}^e, p^e, T)\Delta t^2,$$

$$\|\boldsymbol{u}^e_{\Delta t} - \widetilde{\boldsymbol{u}}_{\Delta t}\|_{\ell^2([H^1(\Omega)]^d)} + \|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^2(L^2(\Omega))} \leq c(\boldsymbol{u}^e, p^e, T)\Delta t^{\frac{3}{2}}.$$

Consistent splitting schemes

- Guermond and Shen (2003) proposed the so-called consistent split scheme.
- The schemes are based on a weak form of the pressure Poisson equation and, at each time step, only require to solve a set of Helmholtz-type equations for the velocity and a Poisson equation (in the weak form) for the pressure.

The key idea

By taking the L^2-inner product of the momentum equation of Stokes equaitons with ∇q, we obtain

$$\int_\Omega \nabla p \cdot \nabla q = \int_\Omega (\boldsymbol{f} + \nu \nabla^2 \boldsymbol{u}) \cdot \nabla q, \quad \forall q \in H^1(\Omega).$$

Assume $\boldsymbol{u}$ is known, then above equation is simply the weak form of a Poisson equation for the pressure.

The principle of the consistent splitting scheme is to compute the velocity and the pressure in two consecutive steps:

1. Compute the velocity by treating the pressure explicitly;
2. Update the pressure using above equation.

Standard splitting scheme

Denote $D := \beta_q \boldsymbol{u}^{n+1} - \sum_{j=0}^{q-1} \beta_j \boldsymbol{u}^{n-j}$. For $n \geq q-1$, find $\boldsymbol{u}^{n+1}$ and p^{n+1} such that

$$\frac{D}{\Delta t}\boldsymbol{u}^{n+1} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1}, \quad \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0,$$
$$(\nabla p^{n+1}, \nabla q) = (\boldsymbol{f}^{n+1} + \nu \nabla^2 \boldsymbol{u}^{n+1}, \nabla q), \quad \forall q \in H^1(\Omega).$$

Note that $\nabla^2 \boldsymbol{u}^{n+1}$ may not be well defined in a finite element discretization. Taking the inner product of the first step with ∇q and subtract the result from the second step, we obtain the following equivalent formulation:

$$\frac{D}{\Delta t}\boldsymbol{u}^{n+1} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1}, \quad \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0,$$
$$(\nabla (p^{n+1} - p^{\star,n+1}), \nabla q) = (\frac{D}{\Delta t}\boldsymbol{u}^{n+1}, \nabla q), \quad \forall q \in H^1(\Omega).$$

Consistent splitting scheme

Theorem (Guermond & Shen (2003))

Provided that the solutions $(\boldsymbol{u}^e_{\Delta t}, p^e_{\Delta t})$ of Stokes problem is smooth enough in time and space, the solution $(u_{\Delta t}, p_{\Delta t})$ of consistent splitting scheme is unconditionally bounded and satisfies the following error estimates:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^2([L^2(\Omega)]^d)} \lesssim \Delta t^2,$$
$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} + \|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} \lesssim \Delta t.$$

Consistent splitting scheme

By Replacing $\nabla^2 \boldsymbol{u}^{n+1}$ by $-\nabla \times \nabla \times \boldsymbol{u}^{n+1}$, leading to the following algorithm:

$$\frac{D}{\Delta t}\boldsymbol{u}^{n+1} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1}, \quad \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0,$$
$$(\nabla p^{n+1}, \nabla q) = (\boldsymbol{f}^{n+1} - \nu \nabla \times \nabla \times \boldsymbol{u}^{n+1}, \nabla q), \quad \forall q \in H^1(\Omega).$$

Again, to avoid computing $-\nabla \times \nabla \times \boldsymbol{u}^{n+1}$. Taking the inner product of the first step with ∇q and subtract the result from the second step:

$$\frac{D}{\Delta t}\boldsymbol{u}^{n+1} - \nu \nabla^2 \boldsymbol{u}^{n+1} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1}, \quad \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0,$$
$$\Big(\nabla \underbrace{(p^{n+1} - p^{\star,n+1} + \nu \nabla \cdot \boldsymbol{u}^{n+1})}_{\phi}, \nabla q\Big) = \Big(\frac{D}{\Delta t}\boldsymbol{u}^{n+1}, \nabla q\Big), \quad \forall q \in H^1(\Omega).$$

Consistent splitting scheme

This leads to an equivalent alternative form:

$$\frac{D}{\Delta t}\boldsymbol{u}^{n+1} - \nu\nabla^2\boldsymbol{u}^{n+1} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1}, \quad \boldsymbol{u}^{n+1}|_{\partial\Omega} = 0,$$

$$(\nabla\phi^{n+1}, \nabla q) = (\frac{D}{\Delta t}\boldsymbol{u}^{n+1}, \nabla q), \quad \forall q \in H^1(\Omega),$$

$$p^{n+1} = \phi^{n+1} + p^{\star,n+1} - \nu\nabla\cdot\boldsymbol{u}^{n+1}.$$

Remark: Neither standard splitting scheme nor consistent splitting is a projection scheme, for the velocity approximtion $\boldsymbol{u}^{n+1}$ is not divergence-free.

Consistent splitting scheme

Theorem (Guermond & Shen (2003))

Provided that the solutions $(\boldsymbol{u}^e_{\Delta t}, p^e_{\Delta t})$ of Stokes problem is smooth enough in time and space, the solution $(u_{\Delta t}, p_{\Delta t})$ of consistent splitting scheme with $q = 1$ is unconditionally bounded and satisfies the following error estimates:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} + \|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} \lesssim \Delta t$$

Conjecture

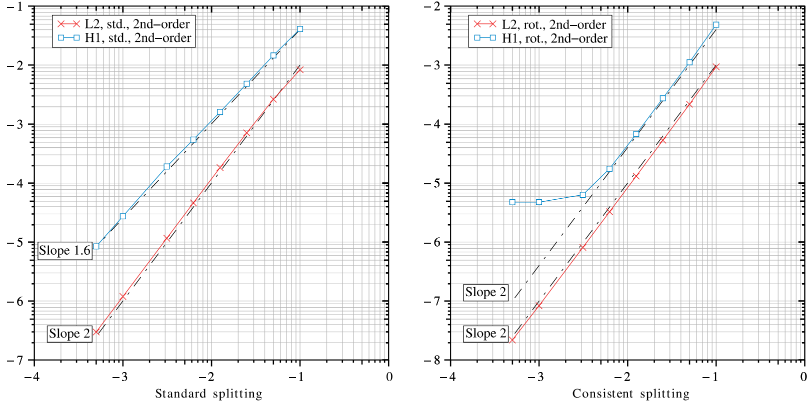
For $q = 2$ the following holds:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} + \|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} \lesssim \Delta t^2$$

The numerical tests seem to confirm the above conjecture.

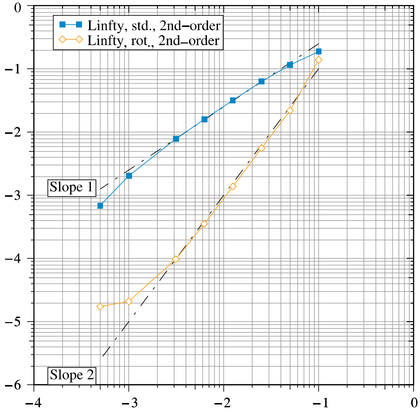
Numerical test (1/2)

Convergence tests using $P_2/P1$ finite element for standard and consistent splitting schemes.



Error on the velocity in the L^2-norm and in the H^1-norm at $T = 1$

Numerical test (2/2)



Error on the pressure in the L^∞-norm at $T = 1$

Relation with the gauge method

- The gauge method was introduced by E and Liu in 2003 to solve an alternative incompressible Navier–Stokes equations.
- The idea is that the pressure is replaced by a so-called gauge variable ξ and define an auxiliary vector field $\boldsymbol{m}$ such that $\boldsymbol{m} = \boldsymbol{u} + \nabla\xi$. Then, the Stokes problem can be reformulated as follows:

$$\frac{\partial \boldsymbol{m}}{\partial t} - \nu\nabla^2 \boldsymbol{m} = \boldsymbol{f}, \quad \boldsymbol{m}\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}, \quad (\boldsymbol{m} - \nabla\xi)\times\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0},$$
$$\nabla^2\xi = \nabla\cdot\boldsymbol{m}, \quad \nabla\xi\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}.$$

The velocity and the pressure are recovered by

$$\boldsymbol{u} = \boldsymbol{m} - \nabla\xi, \quad p = \frac{\partial\xi}{\partial t} - \nu\nabla^2\xi.$$

Time discretization of the gauge method

For $n \geq q-1$, find $\boldsymbol{m}^{n+1}$ and ξ^{n+1} by the following algorithm:

Step 1: Solve for an auxiliary vector field $\boldsymbol{m}^{n+1}$,

$$\frac{D}{\Delta t}\boldsymbol{m}^{n+1} - \nu\nabla^2\boldsymbol{m}^{n+1} = \boldsymbol{f}^{n+1},$$
$$\boldsymbol{m}^{n+1}\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}, \quad (\boldsymbol{m}^{n+1} - \nabla\xi^{\star,n+1})\times\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}.$$

where $\xi^{\star,n+1}$ is the rth order extrapolation for ξ^{n+1} such that $\nabla\xi^{\star,n+1}\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}$.

Step 2: ξ^{n+1} is updated by

$$\nabla^2\xi^{n+1} = \nabla\cdot\boldsymbol{m}^{n+1}, \quad \nabla\xi^{n+1}\cdot\boldsymbol{n}|_{\partial\Omega} = \boldsymbol{0}.$$

Step 3: $\boldsymbol{u}^{n+1}$ and p^{n+1} are recovered by

$$\boldsymbol{u}^{n+1} = \boldsymbol{m}^{n+1} - \nabla\xi^{n+1}, \quad p = \frac{D}{\Delta t}\xi^{n+1} - \nu\nabla^2\xi^{n+1}.$$

Numerical test (E & Liu, 2003)

In this example, E and Liu consider the Navier-Stokes equation with viscosity $\nu = 1$. The finite difference method was used for the spatial discretization and chooses $\Delta t = \Delta x$.

GM1	L_1				L_2				L_∞			
	32^2	64^2	128^2	ord	32^2	64^2	128^2	ord	32^2	64^2	128^2	ord
div $\boldsymbol{u}$	5.01E-3	1.37E-3	3.57E-4	1.91	6.37E-3	1.69E-3	4.37E-4	1.93	1.68E-2	4.98E-3	1.44E-3	1.77
$\boldsymbol{u}$	1.25E-2	7.37E-3	3.99E-3	0.83	1.48E-2	8.57E-3	4.60E-3	0.85	4.10E-2	2.51E-2	1.39E-2	0.79
$\boldsymbol{a}$	7.99E-2	4.33E-2	2.25E-2	0.92	8.89E-2	4.82E-2	2.50E-2	0.92	1.83E-1	1.01E-1	5.28E-2	0.90
ϕ	2.10E-2	1.14E-2	5.91E-3	0.92	2.59E-2	1.41E-2	7.32E-3	0.92	7.41E-2	4.09E-2	2.14E-2	0.90

GM2	L_1				L_2				L_∞			
	32^2	64^2	128^2	ord	32^2	64^2	128^2	ord	32^2	64^2	128^2	ord
div $\boldsymbol{u}$	5.00E-3	1.37E-3	3.57E-4	1.91	6.43E-3	1.71E-3	4.38E-4	1.94	1.87E-2	4.83E-3	1.22E-3	1.97
$\boldsymbol{u}$	2.01E-3	5.25E-4	1.33E-4	1.96	2.34E-3	5.97E-4	1.50E-4	1.98	4.22E-3	1.07E-3	2.68E-4	1.99
$\boldsymbol{a}$	2.57E-3	6.69E-4	1.69E-4	1.97	2.90E-3	7.57E-4	1.91E-4	1.97	7.44E-3	1.99E-3	5.11E-4	1.93
ϕ	8.24E-4	2.14E-4	5.40E-5	1.97	1.03E-3	2.66E-4	6.72E-5	1.97	2.95E-3	7.84E-4	1.99E-4	1.95

In two tables, $\boldsymbol{a} := \boldsymbol{m}$ and $\phi := \xi$

The gauge method and consistent splitting method are "equivalent!"

Staring with the gauge method and changing the variables,

$$\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{m}^{n+1} - \nabla\xi^{\star,n+1}, \qquad \boldsymbol{u}^{n+1} = \boldsymbol{m}^{n+1} - \nabla\xi^{n+1},$$

$$p^{n+1} = \frac{D}{\Delta t}\xi^{n+1} - \nu\nabla^2\xi^{n+1}, \qquad p^{\star,n+1} = \frac{D}{\Delta t}\xi^{\star,n+1} - \nu\nabla^2\xi^{\star,n+1},$$

we can prove that, up to an appropriate change of variables and when the space is continuous, the gauge algorithm is equivalent to the following:

$$\begin{cases} \dfrac{D\widetilde{\boldsymbol{u}}^{n+1}}{\Delta t} - \nu\nabla^2\widetilde{\boldsymbol{u}} + \nabla p^{\star,n+1} = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \widetilde{\boldsymbol{u}} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

$$\begin{cases} \nabla^2\phi^{n+1} = \nabla\cdot\dfrac{D\widetilde{\boldsymbol{u}}^{n+1}}{\Delta t} & \text{in } \Omega, \\ \nabla\phi^{n+1}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega; \end{cases}$$

$$p^{n+1} - p^{\star,n+1} + \nu\nabla\cdot\widetilde{\boldsymbol{u}}^{n+1} = \phi^{n+1}.$$

Summary

Table 1
Stability and convergence rates of rotational schemes[a]

B.C.	q	r	Order	Pressure-correction	Velocity-correction	Consistent splitting
Dirichlet B.C.	1	1	$(1,1)$	Proved	Proved	Proved
	2	1	$\left(2,\frac{3}{2}\right)$	Proved	Proved	Not applicable
	2	2	$(2,2)^{\star}$	FE: stable if $ch^2 \leqslant \Delta t^{\star}$ LG: stable if $cN^{-3} \leqslant \Delta t^{\star}$	FE: stable* LG: stable*	FE: stable* LG: stable*
	3	2	$\left(3,\frac{5}{2}\right)^{\star}$	FE: stable if $ch^2 \leqslant \Delta t^{\star}$ LG: stable if $cN^{-3} \leqslant \Delta t^{\star}$	FE: stable if $ch^2 \leqslant \Delta t^{\star}$ LG: stable*	Not applicable
Open B.C.	1	1	$\left(\frac{3+s}{4},\frac{3+s}{4}\right)$	Proved	Numerical evidences only	Numerical evidences only
	2	1	$\left(\frac{5+s}{4},\frac{3+s}{4}\right)$	Proved	Numerical evidences only	Not applicable
	2	2	Unusable Inf–sup	FE: stable if $c_1h^2 \leqslant \Delta t \leqslant c_2h^{2\star}$ Needed	FE: stable if $c_1h^2 \leqslant \Delta t \leqslant c_2h^{2\star}$ Needed for (4.8) Can be avoided* for (4.6)	FE: stable if $c_1h^2 \leqslant \Delta t \leqslant c_2h^{2\star}$ Needed for (5.10) Can be avoided* for (5.8)

[a] The first (resp. second) number in the parenthesis is the convergence rate for the velocity in the L^2-norm (resp. the velocity in the H^1-norm and the pressure in the L^2-norm); s is the regularity index of the Stokes operator; the symbol $\star$ means numerical evidences only.

Thank you for your attention!