MA 8020: Numerical Analysis II
Numerical Partial Differential Equations

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What are PDEs?

- Most physical phenomena in fluid dynamics, heat transfer, electricity, magnetism, or mechanics can be described in general by partial differential equations (PDEs).
- A PDE is an equation that contains partial derivatives and can be written in the form of

$$F(x_1, x_2, \cdots, x_n, u_{x_1}, u_{x_2}, \cdots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \cdots) = 0.$$

  $u(x_1, x_2, \cdots, x_n)$ is a function of n variables $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$, where u is called the dependent variable and x_i is called the independent variable.

  $u_{x_i} = \dfrac{\partial u}{\partial x_i}$ is the partial derivative of u in the x_i direction.

- A PDE may have one solution, many solutions, or no solution.
- Some constrains are often added to the PDE so that the solution is unique. These are often called boundary or initial conditions.

Kinds of PDEs

- **Linearity:**
  - $F(\cdots) = u_{x_1 x_1} + x_1 u_{x_2 x_2}$ is linear.
  - $F(\cdots) = u_{x_1 x_1} + x_1 u_{x_2 x_2} + u^2$ is nonlinear.
- **Order of the PDEs:** The order of the highest derivative that occurs in F is called the order of the PDE. For example,
  - $u_t = u_{xx}$, second order.
  - $u_t = u u_{xxx} + \sin x$, third order.

Second-order linear equations in two variables

Second-order linear equation in two variables takes a general form of

$$Au_{x_1x_1} + Bu_{x_1x_2} + Cu_{x_2x_2} + Du_{x_1} + Eu_{x_2} + Fu = G.$$

- **Parabolic:** parabolic equations describe heat flow and diffusion processes and satisfy $B^2 - 4AC = 0$. For example,

  heat equation: $u_t = u_{xx}$.

- **Hyperbolic:** hyperbolic equations describe vibrating system and wave motion and satisfy $B^2 - 4AC > 0$. For example,

  wave equation: $u_{tt} = u_{xx}$.

- **Elliptic:** elliptic equations describe steady-state phenomena and satisfy $B^2 - 4AC < 0$. For example,

  Poisson's equation: $-(u_{xx} + u_{yy}) = f(x, y)$.

Application of Poisson's equation in heat transfer

Let Ω be an open and bounded domain with boundary $\partial\Omega$. Consider

$$-(u_{x_1x_1} + u_{x_2x_2}) = f(x_1, x_2) \quad \text{on } \Omega$$

is used for describing steady state temperature distribution of some material. Three types of boundary conditions:

- **Dirichlet condition:** $u = g(s)$ on $\partial\Omega$, temperature specified on the boundary.
- **Neumann condition:** $\dfrac{\partial u}{\partial \boldsymbol{n}} = h(s)$ on $\partial\Omega$, $\boldsymbol{n}$ is an outward unit normal vector, heat flow across the boundary (flux) specified.

  Note that $\dfrac{\partial u}{\partial \boldsymbol{n}} = \nabla u \cdot \boldsymbol{n}$.
- **Mixed condition:** $\dfrac{\partial u}{\partial \boldsymbol{n}} + \lambda u = g(s)$ on $\partial\Omega$.

1-D Heat equation

- Initial-boundary value problem (IBVP): find $u(x,t)$ such that

$$\begin{cases} u_t = u_{xx} & t > 0, 0 < x < 1, \\ u(x,0) = g(x) & 0 \le x \le 1, \\ u(0,t) = a(t) & t \ge 0, \\ u(1,t) = b(t) & t \ge 0. \end{cases}$$

- **Notations:** $u(x,t)$: unknown temperature in the rod, x is spatial coordinates and t is time, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ and $u_t = \dfrac{\partial u}{\partial t}$.

Finite difference method

- Let
$$\begin{cases} t_j = jk & j \ge 0, \\ x_i = ih & 0 \le i \le n+1. \end{cases}$$
Note that $k \neq h$ in general.

- Recall some finite difference approximations:
$$\begin{aligned} f'(x) &\approx \frac{1}{h}\Big(f(x+h) - f(x)\Big), \\ f'(x) &\approx \frac{1}{2h}\Big(f(x+h) - f(x-h)\Big), \\ f''(x) &\approx \frac{1}{h^2}\Big(f(x+h) - 2f(x) + f(x-h)\Big). \end{aligned}$$

Finite difference method: explicit method

- Let $v \approx u$. Then

$$\frac{1}{k}\Big(v(x,t+k) - v(x,t)\Big) = \frac{1}{h^2}\Big(v(x+h,t) - 2v(x,t) + v(x-h,t)\Big).$$

By defining $v_{ij} = v(x_i, t_j)$, we have

$$\frac{1}{k}\Big(v_{i,j+1} - v_{i,j}\Big) = \frac{1}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big).$$

- Rewrite the above equation to obtain

$$v_{i,j+1} = \frac{k}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big) + v_{i,j}$$

or

$$v_{i,j+1} = \Big(sv_{i-1,j} + (1-2s)v_{i,j} + sv_{i+1,j}\Big),$$

with $s = k/h^2$.

Algorithm

input n , k , M

$h \leftarrow \frac{1}{n+1}$ and $s \leftarrow \frac{k}{h^2}$

$w_i = g(ih)\ (0 \le i \le n+1)$

$t \leftarrow 0$

output $0, t, (w_0, w_1, \cdots, w_{n+1})$

for j = 1 **to** M **do**

$v_0 \leftarrow a(jk)$ and $v_{n+1} \leftarrow b(jk)$

  for i = 1 **to** n **do**

  $v_i = \left(sw_{i-1} + (1-2s)w_i + sw_{i+1}\right)$

  end do

  $t \leftarrow jk$

  output j , t , $(v_0, v_1, \cdots, v_{n+1})$

  $(w_1, w_2, \cdots, w_n) \leftarrow (v_1, v_2, \cdots, v_n)$

end do

Stability analysis

- Assume that $a(t) = b(t) = 0$. At $t_j = jk$, define $V_j = (v_{1,j}, v_{2,j}, \cdots, v_{n,j})^\top$. Then the explicit difference equations becomes $V_{j+1} = AV_j$, where

$$A = \begin{bmatrix} 1-2s & s & & & & \\ s & 1-2s & s & & & \\ & s & 1-2s & s & & \\ & & \ddots & \ddots & \ddots & \\ & & & s & 1-2s & s \\ & & & & s & 1-2s \end{bmatrix}.$$

- Note that $v_{0,j} = v_{n+1,j} = 0$. We know that exact solution approaches 0 as $t \to \infty$ and therefore the temperature will reduce to zero as $t \to \infty$.

Stability analysis (cont'd)

- For the numerical approximation,

$$V_{j+1} = AV_j = A(AV_{j-1}) = \cdots = A^{j+1}V_0.$$

- Recall the following two statements are equivalent (see Section 4.6, p. 215)
  (1) $\lim_{j\to\infty} A^j V = 0$ *for all vectors* $V \in \mathbb{R}^n$.
  (2) $\rho(A) < 1$, *where* $\rho(A)$ *is the spectral radius of matrix* A.
- So $s = k/h^2$ should be chosen such that $\rho(A) < 1$.

  The eigenvalues of A are: $\lambda_j = 1 - 2s(1 - \cos\theta_j)$, where $\theta_j = \frac{j\pi}{n+1}$, $1 \le j \le n$.

  For $\rho(A) < 1$ we require $-1 < 1 - 2s(1 - \cos\theta_j) < 1$.

  This is true if and only if $s < (1 - \cos\theta_j)^{-1}$.

Stability analysis (cont'd)

- The greatest restriction on s occurs when $\cos\theta_j = -1$, which does not happen since the largest $\theta_{j=n} = \frac{n\pi}{n+1}$. So we have $0 < s \le 1/2$ (i.e., $k/h^2 \le 1/2$) $\Longleftrightarrow k \le \frac{h^2}{2}$.
- For example, $h = 0.01 \Longrightarrow k \le 5 \times 10^{-5} \Rightarrow$ For $0 \le t \le 10$, the number of time step: 0.5×10^6.
- **Open question:** Find eigenvalue of A. Note $A = I - sB$, where

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

If x_i is an eigenvector of B with eigenvalue μ_i then

$$(I - sB)x_i = x_i - s\mu_i x_i = (1 - s\mu_i)x_i = Ax_i.$$

Hence $\lambda_i = 1 - s\mu_i$ is an eigenvalue of A.

Lemma on tridiagonal matrix eigenvalues and eigenvectors

Let $x = (\sin\theta, \sin 2\theta, \cdots, \sin n\theta)^\top$*. If* $\theta = \frac{j\pi}{n+1}$*, then* x *is an eigenvector of* B *corresponding to the eigenvalue* $2 - 2\cos\theta$*.*

Proof: see textbook, page 621.

Finite difference method: implicit method

- We continue to study the initial-boundary value problem: find $u(x,t)$ such that

$$\begin{cases} u_t &= u_{xx} \quad t > 0, 0 < x < 1, \\ u(x,0) &= g(x) \quad 0 \le x \le 1, \\ u(0,t) &= 0 \quad t \ge 0, \\ u(1,t) &= 0 \quad t \ge 0. \end{cases}$$

- The finite-difference equation :

$$\frac{1}{k}\Big(v(x,t) - v(x,t-k)\Big) = \frac{1}{h^2}\Big(v(x+h,t) - 2v(x,t) + v(x-h,t)\Big).$$

$$\Longrightarrow \frac{1}{k}\Big(v_{i,j} - v_{i,j-1}\Big) = \frac{1}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big).$$

- Let $s = \frac{k}{h^2}$ and rearrange to obtain

$$-sv_{i+1,j} + (1+2s)v_{i,j} - sv_{i-1,j} = v_{i,j-1}, \text{ for } 1 \le i \le n.$$

Stability analysis

- Let $V_j = (v_{1,j}, v_{2,j}, \cdots, v_{n,j})^\top$ then the method can be written as $AV_j = V_{j-1}$, where A is given by

$$A = \begin{bmatrix} 1+2s & -s & & & \\ -s & 1+2s & -s & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & -s & 1+2s \end{bmatrix}.$$

- Solve $V_j = A^{-1}V_{j-1} = A^{-1}A^{-1}V_{j-2} \cdots = A^{-j}V_0$.
- V_0 is known, $u(ih, 0)$ initial condition. Here we need $\rho(A^{-1}) < 1$ for stability.

Stability analysis (cont'd)

- Since $A = I + sB$, where

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix},$$

  the eigenvalues of A are given by $\lambda_i = 1 + s\mu_i$ $= 1 + 2s(1 - \cos\theta_i)$ with $\theta_i = \frac{i\pi}{n+1}$, $1 \le i \le n$.

- Clearly, $\lambda_i > 1$, since $\lambda_i = 1 + 2s(1 - \cos\theta_i)$

  $\Longrightarrow \lambda_i > 1 \Rightarrow \rho(A^{-1}) < 1$.

  $\Longrightarrow$ *The method is stable for all h and k.*

- Note that we need to solve a tridiagonal system of linear equation to advance each time step (use subroutine §4.3 to solve tridiagonal).

Algorithm

input n, k, M

$h \leftarrow \frac{1}{n+1}, \quad s \leftarrow \frac{k}{h^2}, \quad v_i = g(ih)\ (1 \le i \le n), \quad t \leftarrow 0$

output $0, t, (v_1, v_2, \cdots, v_n)$

for $i = 1$ **to** $n - 1$ **do**

  $c_i = -s$ and $a_i = -s$

end do

for $j = 1$ **to** M **do**

  for $i = 1$ **to** n **do**

    $d_i = 1 + 2s$

  end do

  call tri$(n, a, d, c, v; v)$

  $t \leftarrow jk$

  output $j, t, (v_1, v_2, \cdots, v_n)$

end do

Crank-Nicolson method

We can combine the previous two methods into a θ-method:

$$\frac{\theta}{h^2}\left(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\right) + \frac{1-\theta}{h^2}\left(v_{i+1,j-1} - 2v_{i,j-1} + v_{i-1,j-1}\right)$$
$$= \frac{1}{k}\left(v_{i,j} - v_{i,j-1}\right).$$

- $\theta = 0 \Longrightarrow$ explicit method.
- $\theta = 1 \Longrightarrow$ implicit method.
- $\theta = 1/2 \Longrightarrow$ Crank-Nicolson (CN).

Crank-Nicolson method (cont'd)

- Taking $s = \frac{k}{h^2}$ and rewriting the CN method, we obtain

$$-sv_{i-1,j} + (2+2s)v_{i,j} - sv_{i+1,j} = sv_{i-1,j-1} + (2+2s)v_{i,j-1} + sv_{i+1,j-1}.$$

- Again, let $V_j = (v_{1,j}, v_{2,j}, \cdots, v_{n,j})^\top$ and

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix}.$$

The method can be written in the matrix form

$$(2I + sB)V_j = (2I - sB)V_{j-1}.$$

Stability analysis

- For stability, we need $\rho((2I + sB)^{-1}(2I - sB)) < 1$.
- Set $A = (2I + sB)^{-1}(2I - sB)$ with $V_j = AV_{j-1}$. If x_i is an eigenvector of B then

$$\begin{aligned}(2I - sB)x_i &= 2x_i - sBx_i \\ &= 2x_i - s\mu_i x_i \\ &= (2 - s\mu_i)x_i. \\ (2I + sB)^{-1}(2I - sB)x_i &= (2I + sB)^{-1}(2 - s\mu_i)x_i = \cdots\end{aligned}$$

$\Longrightarrow x_i$ is also an eigenvector of A with eigenvalues $\frac{2-s\mu_i}{2+s\mu_i}$.

- To have $\rho((2 + sB)^{-1}(2 - sB)) < 1$, we get it if $|(2 + s\mu)^{-1}(2 - s\mu)| < 1$.

  Because $\mu_i = 2(1 - \cos\theta_i)$, we see that $0 < \mu_i < 4$.

  Thus $|\frac{2-s\mu_i}{2+s\mu_i}| < 1, \forall s = \frac{k}{h^2}$.

  So, the CN method is an unconditionally stable method.

Error analysis

- Recall the explicit method

$$v_{i,j+1} = s(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}) + v_{i,j}.$$

Let $u_{i,j}$ be the exact solution at (x_i, t_j). Then

$$e_{i,j} = u_{i,j} - v_{i,j}.$$

- We replace v by $u - e$ to obtain

$$\begin{aligned} u_{i,j+1} - e_{i,j+1} &= s(u_{i-i,j} - 2u_{i,j} + u_{i+1,j}) + u_{i,j} \\ &\quad - s(e_{i-i,j} - 2e_{i,j} + e_{i+1,j}) - e_{i,j}. \\ e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) \\ &\quad - s(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (u_{i,j+1} - u_{i,j}). \end{aligned}$$

Error analysis (cont'd)

Using these formulas,

$$
\begin{aligned}
f''(x) &= \frac{1}{h^2}\Big(f(x+h) - 2f(x) + f(x-h)\Big) - \frac{h^2}{12} f^{(4)}(\xi), \\
g'(t) &= \frac{1}{k}\Big(g(t+k) - g(t)\Big) - \frac{k}{2} g''(\tau),
\end{aligned}
$$

with $sh^2 = k$ and $u_{xx} = u_t$, we obtain

$$
\begin{aligned}
e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) - s(h^2 u_{xx}(x_i, t_j) + \frac{h^4}{12} u_{xxxx}(\xi_i, t_j)) \\
&\quad + (ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, \tau_j)), \\
\Rightarrow e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) \\
&\quad - kh^2 (\frac{1}{12} u_{xxxx}(\xi_i, t_j) - \frac{s}{2} u_{tt}(x_i, \tau_i)).
\end{aligned}
$$

Error analysis (cont'd)

- Let us confine (x,t) to the compact set

$$S = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}.$$

Put $M = \frac{1}{12} \max_S |u_{xxxx}(x,t)| + \frac{1}{2} \max_S |u_{tt}(x,t)|$,
$E_j = (e_{1,j}, e_{2,j}, \cdots, e_{n,j})^\top$, $\|E_j\|_\infty = \max_{1 \le i \le n} |e_{ij}|$.

- We assume $1 - 2s > 0$:

$$\begin{aligned} |e_{i,j+1}| &\le s|e_{i-1,j}| + (1-2s)|e_{ij}| + s|e_{i+1,j}| + kh^2 M \\ &\le s\|E_j\|_\infty + (1-2s)\|E_j\|_\infty + s\|E_j\|_\infty + kh^2 M \\ &\le \|E_j\|_\infty + kh^2 M. \end{aligned}$$

Hence,

$$\begin{aligned} \|E_{j+1}\|_\infty &\le \|E_j\|_\infty + kh^2 M \le \|E_{j-1}\|_\infty + 2kh^2 M \\ &\le \cdots \le \|E_0\|_\infty + (j+1)kh^2 M. \end{aligned}$$

$$\Longrightarrow \|E_j\|_\infty \le jkh^2 M \Longrightarrow \|E_j\|_\infty \le Th^2 M = O(h^2).$$

Numerical differentiation

Assume that $u \in C^4[a,b]$ and $a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b$ is a uniform partition of $[a,b]$. Then $h_j = h = \frac{b-a}{M+1}$ for $j = 1, 2, \cdots, M+1$. For $i = 1, 2, \cdots, M$, we have

$$u(x_i + h) = u(x_i) + u'(x_i)h + \frac{1}{2}u''(x_i)h^2 + \frac{1}{6}u^{(3)}(x_i)h^3 + \frac{1}{24}u^{(4)}(\xi_{i1})h^4,$$
$$u(x_i - h) = u(x_i) - u'(x_i)h + \frac{1}{2}u''(x_i)h^2 - \frac{1}{6}u^{(3)}(x_i)h^3 + \frac{1}{24}u^{(4)}(\xi_{i2})h^4,$$

for some $\xi_{i1} \in (x_i, x_i + h)$ and $\xi_{i2} \in (x_i - h, x_i)$. Therefore, we have

$$u(x_i + h) + u(x_i - h) = 2u(x_i) + u''(x_i)h^2 + \frac{1}{24}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}h^4.$$

$$\therefore u''(x_i) =$$
$$\frac{1}{h^2}\{u(x_i + h) - 2u(x_i) + u(x_i - h)\} - \frac{1}{24}h^2\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$$

$\because u \in C^4[a,b]$, $\frac{1}{2}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}$ between $u^{(4)}(\xi_{i1})$ & $u^{(4)}(\xi_{i2})$.

$\therefore$ By IVT, $\exists\ \xi_i$ between ξ_{i1} and ξ_{i2} ($\Rightarrow \xi_i \in (x_i - h, x_i + h)$) such that

$$u^{(4)}(\xi_i) = \frac{1}{2}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$$

$$\therefore u''(x_i) = \frac{1}{h^2}\{u(x_i + h) - 2u(x_i) + u(x_i - h)\} - \frac{1}{12}h^2 u^{(4)}(\xi_i),$$

for some $\xi_i \in (x_i - h, x_i + h)$. (2nd-order approximation)

FDM for a two-point boundary value problem

- Consider the 1-D two-point BVP:

$$\begin{cases} -u''(x) & = & f(x) \quad x \in (0,1), \\ u(0) & = & u(1) = 0. \end{cases}$$

- The interval $[0,1]$ is discretized uniformly by taking the $n+2$ points, $x_i = ih$, for $i = 0, 1, \cdots, n+1$, where $h = 1/(n+1)$.
- Let $v_i \approx u(x_i)$, $i = 1, 2, \cdots, n$, and $v_0 := u(x_0) = 0$, $v_{n+1} := u(x_{n+1}) = 0$ are known due to the Dirichlet BC.
- If the centered difference approximation is used for u'', the above equation can be expressed as

$$-\left(\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}\right) = f_i, \quad i = 1, 2, \cdots, n,$$

where $f_i := f(x_i)$.

Resulting linear system

The linear system obtained is of the form

$$AV = F,$$

where

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

$$V = (v_1, v_2, \cdots, v_n)^\top \quad \text{and} \quad F = (h^2 f_1, h^2 f_2, \cdots, h^2 f_n)^\top.$$

Eigen properties of A

- The matrix A has n eigenvalues, and since A is symmetric, all eigenvalues must be real.
- Note that the eigenvalues of A are given by

$$\lambda_j = 2 - 2\cos(j\theta) > 0, j = 1, 2, \cdots, n,$$

and the eigenvector associated with each λ_j is given by

$$V_j = (\sin(j\theta), \sin(2j\theta), \cdots, \sin(nj\theta))^\top,$$

where $\theta = \frac{\pi}{n+1}$.

- $\lambda_{\max} = 2 - 2\cos(\frac{n\pi}{n+1})$ and $\lambda_{\min} = 2 - 2\cos(\frac{\pi}{n+1})$.
- What is the condition number of A?
  $(\sin x = x - x^3/(3!) + x^5/(5!) \cdots)$

$$\kappa(A) = \frac{\sin^2 \frac{n\pi}{2(n+1)}}{\sin^2 \frac{\pi}{2(n+1)}} \approx \frac{\sin(\pi/2)}{(\frac{\pi}{2(n+1)})^2} \approx O\left(n^2\right) \approx O\left(\frac{1}{h^2}\right).$$

FDM for a 2-D boundary value problem

- Consider Poisson's problem,

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = f & \text{in } \Omega := (0,1)\times(0,1), \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- Define the mesh size $h = \frac{1}{n+1}$, the collection of mesh points $(x_{1i}, x_{2j}) = (ih, jh)$, the approximate solution at the mesh points $v_{ij} \approx u(x_{1j}, x_{2j})$, $i, j = 0, 1, \cdots, n+1$.

  Note: There are n^2 interior points $\approx \frac{1}{h^2}$. (in 3D, $\approx \frac{1}{h^3}$ number of points).

- The FD equations

$$\begin{cases} \dfrac{v_{i-1j} - 2v_{ij} + v_{i+1j}}{h^2} + \dfrac{v_{ij-1} - 2v_{ij} + v_{ij+1}}{h^2} = f_{ij}, \\ v_{0j} = v_{n+1j} = v_{i0} = v_{in+1} = 0. \end{cases}$$

For example $n = 3$: natural ordering

- We order the unknown quantities in the natural ordering

$$V = (v_{11}, v_{21}, v_{31}, v_{12}, v_{22}, v_{n2}, v_{13}, v_{23}, v_{33})^\top.$$

- Then the corresponding linear system can be written as (see Text, page 631)

$$AV = \begin{bmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{bmatrix} V = F \quad \text{with} \quad B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

- block-tridiag matrix; symmetric $a_{ij} = a_{ji}$; sparse, number of nonzeros per row ≈ 5 (independent of the mesh size h) number of nonzeros $\approx 5n$ (linear in n).

References

- K. W. Morton and D. F. Mayers, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, 1994.
- K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman & Hall, 1996.