

# MA 8020: Numerical Analysis II

## Numerical Partial Differential Equations



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## What are PDEs?

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- Most physical phenomena in fluid dynamics, heat transfer, electricity, magnetism, or mechanics can be described in general by partial differential equations (PDEs).
- A PDE is an equation that contains partial derivatives and can be written in the form of

$$F(x_1, x_2, \dots, x_n, u_{x_1}, u_{x_2}, \dots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \dots) = 0.$$

$u(x_1, x_2, \dots, x_n)$  is a function of  $n$  variables  
 $x = (x_1, x_2, \dots, x_n)^\top \in \mathbb{R}^n$ , where  $u$  is called the dependent variable and  $x_i$  is called the independent variable.

$u_{x_i} = \frac{\partial u}{\partial x_i}$  is the partial derivative of  $u$  in the  $x_i$  direction.

- A PDE may have one solution, many solutions, or no solution.
- Some constraints are often added to the PDE so that the solution is unique. These are often called boundary or initial conditions.

## Kinds of PDEs

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- **Linearity:**

- $F(\cdots) = u_{x_1x_1} + x_1u_{x_2x_2}$  is linear.
- $F(\cdots) = u_{x_1x_1} + x_1u_{x_2x_2} + u^2$  is nonlinear.

- **Order of the PDEs:** The order of the highest derivative that occurs in  $F$  is called the order of the PDE. For example,

- $u_t = u_{xx}$ , second order.
- $u_t = uu_{xxx} + \sin x$ , third order.

## Second-order linear equations in two variables

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Second-order linear equation in two variables takes a general form of

$$Au_{x_1x_1} + Bu_{x_1x_2} + Cu_{x_2x_2} + Du_{x_1} + Eu_{x_2} + Fu = G.$$

- **Parabolic:** parabolic equations describe heat flow and diffusion processes and satisfy  $B^2 - 4AC = 0$ . For example,  
heat equation:  $u_t = u_{xx}$ .
- **Hyperbolic:** hyperbolic equations describe vibrating system and wave motion and satisfy  $B^2 - 4AC > 0$ . For example,  
wave equation:  $u_{tt} = u_{xx}$ .
- **Elliptic:** elliptic equations describe steady-state phenomena and satisfy  $B^2 - 4AC < 0$ . For example,  
Poisson's equation:  $-(u_{xx} + u_{yy}) = f(x, y)$ .

## Application of Poisson's equation in heat transfer

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Let  $\Omega$  be an open and bounded domain with boundary  $\partial\Omega$ . Consider

$$-(u_{x_1x_1} + u_{x_2x_2}) = f(x_1, x_2) \quad \text{on } \Omega$$

is used for describing steady state temperature distribution of some material. Three types of boundary conditions:

- **Dirichlet condition:**  $u = g(s)$  on  $\partial\Omega$ , temperature specified on the boundary.
- **Neumann condition:**  $\frac{\partial u}{\partial \mathbf{n}} = h(s)$  on  $\partial\Omega$ ,  $\mathbf{n}$  is an outward unit normal vector, heat flow across the boundary (flux) specified.

Note that  $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ .

- **Mixed condition:**  $\frac{\partial u}{\partial \mathbf{n}} + \lambda u = g(s)$  on  $\partial\Omega$ .

## 1-D Heat equation

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- Initial-boundary value problem (IBVP): find  $u(x, t)$  such that

$$\begin{cases} u_t = u_{xx} & t > 0, 0 < x < 1, \\ u(x, 0) = g(x) & 0 \leq x \leq 1, \\ u(0, t) = a(t) & t \geq 0, \\ u(1, t) = b(t) & t \geq 0. \end{cases}$$

- Notations:**  $u(x, t)$ : unknown temperature in the rod,  $x$  is spatial coordinates and  $t$  is time,  $u_{xx} = \frac{\partial^2 u}{\partial x^2}$  and  $u_t = \frac{\partial u}{\partial t}$ .

## Finite difference method

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- Let

$$\begin{cases} t_j &= jk & j \geq 0, \\ x_i &= ih & 0 \leq i \leq n+1. \end{cases}$$

Note that  $k \neq h$  in general.

- Recall some finite difference approximations:

$$f'(x) \approx \frac{1}{h} (f(x+h) - f(x)),$$

$$f'(x) \approx \frac{1}{2h} (f(x+h) - f(x-h)),$$

$$f''(x) \approx \frac{1}{h^2} (f(x+h) - 2f(x) + f(x-h)).$$

## Finite difference method: explicit method

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- Let  $v \approx u$ . Then

$$\frac{1}{k} \left( v(x, t+k) - v(x, t) \right) = \frac{1}{h^2} \left( v(x+h, t) - 2v(x, t) + v(x-h, t) \right).$$

By defining  $v_{ij} = v(x_i, t_j)$ , we have

$$\frac{1}{k} \left( v_{i,j+1} - v_{i,j} \right) = \frac{1}{h^2} \left( v_{i+1,j} - 2v_{i,j} + v_{i-1,j} \right).$$

- Rewrite the above equation to obtain

$$v_{i,j+1} = \frac{k}{h^2} \left( v_{i+1,j} - 2v_{i,j} + v_{i-1,j} \right) + v_{i,j}$$

or

$$v_{i,j+1} = \left( sv_{i-1,j} + (1-2s)v_{i,j} + sv_{i+1,j} \right),$$

with  $s = k/h^2$ .



## Algorithm

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**input**  $n, k, M$

$h \leftarrow \frac{1}{n+1}$  and  $s \leftarrow \frac{k}{h^2}$

$w_i = g(ih)$  ( $0 \leq i \leq n+1$ )

$t \leftarrow 0$

**output**  $0, t, (w_0, w_1, \dots, w_{n+1})$

**for**  $j = 1$  **to**  $M$  **do**

$v_0 \leftarrow a(jk)$  and  $v_{n+1} \leftarrow b(jk)$

**for**  $i = 1$  **to**  $n$  **do**

$v_i = (sw_{i-1} + (1 - 2s)w_i + sw_{i+1})$

**end do**

$t \leftarrow jk$

**output**  $j, t, (v_0, v_1, \dots, v_{n+1})$

$(w_1, w_2, \dots, w_n) \leftarrow (v_1, v_2, \dots, v_n)$

**end do**

## Stability analysis

- Assume that  $a(t) = b(t) = 0$ . At  $t_j = jk$ , define  $V_j = (v_{1,j}, v_{2,j}, \dots, v_{n,j})^\top$ . Then the explicit difference equations becomes  $V_{j+1} = AV_j$ , where

$$A = \begin{bmatrix} 1-2s & s & & & & \\ s & 1-2s & s & & & \\ & s & 1-2s & s & & \\ & & \ddots & \ddots & \ddots & \\ & & & s & 1-2s & s \\ & & & & s & 1-2s \end{bmatrix}.$$

- Note that  $v_{0,j} = v_{n+1,j} = 0$ . We know that exact solution approaches 0 as  $t \rightarrow \infty$  and therefore the temperature will reduce to zero as  $t \rightarrow \infty$ .

## Stability analysis (cont'd)

- For the numerical approximation,

$$V_{j+1} = AV_j = A(AV_{j-1}) = \cdots = A^{j+1}V_0.$$

- Recall the following two statements are equivalent (see Section 4.6, p. 215)

(1)  $\lim_{j \rightarrow \infty} A^j V = 0$  for all vectors  $V \in \mathbb{R}^n$ .

(2)  $\rho(A) < 1$ , where  $\rho(A)$  is the spectral radius of matrix  $A$ .

- So  $s = k/h^2$  should be chosen such that  $\rho(A) < 1$ .

The eigenvalues of  $A$  are:  $\lambda_j = 1 - 2s(1 - \cos \theta_j)$ , where  $\theta_j = \frac{j\pi}{n+1}$ ,  $1 \leq j \leq n$ .

For  $\rho(A) < 1$  we require  $-1 < 1 - 2s(1 - \cos \theta_j) < 1$ .

This is true if and only if  $s < (1 - \cos \theta_j)^{-1}$ .

## Stability analysis (cont'd)

- The greatest restriction on  $s$  occurs when  $\cos \theta_j = -1$ , which does not happen since the largest  $\theta_{j=n} = \frac{n\pi}{n+1}$ . So we have  $0 < s \leq 1/2$  (i.e.,  $k/h^2 \leq 1/2$ )  $\iff k \leq \frac{h^2}{2}$ .
- For example,  $h = 0.01 \implies k \leq 5 \times 10^{-5} \Rightarrow$  For  $0 \leq t \leq 10$ , the number of time step:  $0.5 \times 10^6$ .
- **Open question:** Find eigenvalue of  $A$ . Note  $A = I - sB$ , where

$$B = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

If  $x_i$  is an eigenvector of  $B$  with eigenvalue  $\mu_i$  then

$$(I - sB)x_i = x_i - s\mu_i x_i = (1 - s\mu_i)x_i = Ax_i.$$

Hence  $\lambda_i = 1 - s\mu_i$  is an eigenvalue of  $A$ .

## Lemma on tridiagonal matrix eigenvalues and eigenvectors

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*Let  $x = (\sin \theta, \sin 2\theta, \dots, \sin n\theta)^\top$ . If  $\theta = \frac{j\pi}{n+1}$ , then  $x$  is an eigenvector of  $B$  corresponding to the eigenvalue  $2 - 2 \cos \theta$ .*

*Proof:* see textbook, page 621.

## Finite difference method: implicit method

- We continue to study the initial-boundary value problem: find  $u(x, t)$  such that

$$\begin{cases} u_t = u_{xx} & t > 0, 0 < x < 1, \\ u(x, 0) = g(x) & 0 \leq x \leq 1, \\ u(0, t) = 0 & t \geq 0, \\ u(1, t) = 0 & t \geq 0. \end{cases}$$

- The finite-difference equation :

$$\begin{aligned} \frac{1}{k} \left( v(x, t) - v(x, t - k) \right) &= \frac{1}{h^2} \left( v(x + h, t) - 2v(x, t) + v(x - h, t) \right). \\ \implies \frac{1}{k} \left( v_{i,j} - v_{i,j-1} \right) &= \frac{1}{h^2} \left( v_{i+1,j} - 2v_{i,j} + v_{i-1,j} \right). \end{aligned}$$

- Let  $s = \frac{k}{h^2}$  and rearrange to obtain

$$-sv_{i+1,j} + (1 + 2s)v_{i,j} - sv_{i-1,j} = v_{i,j-1}, \text{ for } 1 \leq i \leq n.$$

## Stability analysis

- Let  $V_j = (v_{1,j}, v_{2,j}, \dots, v_{n,j})^\top$  then the method can be written as  $AV_j = V_{j-1}$ , where  $A$  is given by

$$A = \begin{bmatrix} 1+2s & -s & & & \\ -s & 1+2s & -s & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -s & 1+2s \end{bmatrix}.$$

- Solve  $V_j = A^{-1}V_{j-1} = A^{-1}A^{-1}V_{j-2} \dots = A^{-j}V_0$ .
- $V_0$  is known,  $u(ih, 0)$  initial condition. Here we need  $\rho(A^{-1}) < 1$  for stability.

## Stability analysis (cont'd)

- Since  $A = I + sB$ , where

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -1 & 2 \end{bmatrix},$$

the eigenvalues of  $A$  are given by  $\lambda_i = 1 + s\mu_i$   
 $= 1 + 2s(1 - \cos \theta_i)$  with  $\theta_i = \frac{i\pi}{n+1}$ ,  $1 \leq i \leq n$ .

- Clearly,  $\lambda_i > 1$ , since  $\lambda_i = 1 + 2s(1 - \cos \theta_i)$

$$\implies \lambda_i > 1 \implies \rho(A^{-1}) < 1.$$

*$\implies$  The method is stable for all  $h$  and  $k$ .*

- Note that we need to solve a tridiagonal system of linear equation to advance each time step (use subroutine §4.3 to solve tridiagonal).



## Algorithm

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**input**  $n, k, M$

$h \leftarrow \frac{1}{n+1}, \quad s \leftarrow \frac{k}{h^2}, \quad v_i = g(ih) \quad (1 \leq i \leq n), \quad t \leftarrow 0$

**output**  $0, t, (v_1, v_2, \dots, v_n)$

**for**  $i = 1$  **to**  $n - 1$  **do**

$c_i = -s$  and  $a_i = -s$

**end do**

**for**  $j = 1$  **to**  $M$  **do**

**for**  $i = 1$  **to**  $n$  **do**

$d_i = 1 + 2s$

**end do**

call tri( $n, a, d, c, v; v$ )

$t \leftarrow jk$

**output**  $j, t, (v_1, v_2, \dots, v_n)$

**end do**

## Crank-Nicolson method

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We can combine the previous two methods into a  $\theta$ -method:

$$\begin{aligned} & \frac{\theta}{h^2} \left( v_{i+1,j} - 2v_{i,j} + v_{i-1,j} \right) + \frac{1-\theta}{h^2} \left( v_{i+1,j-1} - 2v_{i,j-1} + v_{i-1,j-1} \right) \\ &= \frac{1}{k} \left( v_{i,j} - v_{i,j-1} \right). \end{aligned}$$

- $\theta = 0 \implies$  explicit method.
- $\theta = 1 \implies$  implicit method.
- $\theta = 1/2 \implies$  Crank-Nicolson (CN).

## Crank-Nicolson method (cont'd)

- Taking  $s = \frac{k}{h^2}$  and rewriting the CN method, we obtain

$$-sv_{i-1,j} + (2 + 2s)v_{i,j} - sv_{i+1,j} = sv_{i-1,j-1} + (2 + 2s)v_{i,j-1} + sv_{i+1,j-1}.$$

- Again, let  $V_j = (v_{1,j}, v_{2,j}, \dots, v_{n,j})^\top$  and

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -1 & 2 \end{bmatrix}.$$

The method can be written in the matrix form

$$(2I + sB)V_j = (2I - sB)V_{j-1}.$$

## Stability analysis

- For stability, we need  $\rho((2I + sB)^{-1}(2I - sB)) < 1$ .
- Set  $A = (2I + sB)^{-1}(2I - sB)$  with  $V_j = AV_{j-1}$ . If  $x_i$  is an eigenvector of  $B$  then

$$\begin{aligned}(2I - sB)x_i &= 2x_i - sBx_i \\ &= 2x_i - s\mu_i x_i \\ &= (2 - s\mu_i)x_i.\end{aligned}$$

$$(2I + sB)^{-1}(2I - sB)x_i = (2I + sB)^{-1}(2 - s\mu_i)x_i = \cdots$$

$\implies x_i$  is also an eigenvector of  $A$  with eigenvalues  $\frac{2-s\mu_i}{2+s\mu_i}$ .

- To have  $\rho((2I + sB)^{-1}(2I - sB)) < 1$ , we get it if  $|(2 + s\mu)^{-1}(2 - s\mu)| < 1$ .

Because  $\mu_i = 2(1 - \cos \theta_i)$ , we see that  $0 < \mu_i < 4$ .

Thus  $|\frac{2-s\mu_i}{2+s\mu_i}| < 1, \forall s = \frac{k}{h^2}$ .

So, the CN method is an unconditionally stable method.

## Error analysis

- Recall the explicit method

$$v_{i,j+1} = s(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}) + v_{i,j}.$$

Let  $u_{i,j}$  be the exact solution at  $(x_i, t_j)$ . Then

$$e_{i,j} = u_{i,j} - v_{i,j}.$$

- We replace  $v$  by  $u - e$  to obtain

$$\begin{aligned}u_{i,j+1} - e_{i,j+1} &= s(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + u_{i,j} \\&\quad - s(e_{i-1,j} - 2e_{i,j} + e_{i+1,j}) - e_{i,j}. \\e_{i,j+1} &= (se_{i-1,j} + (1 - 2s)e_{i,j} + se_{i+1,j}) \\&\quad - s(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (u_{i,j+1} - u_{i,j}).\end{aligned}$$

## Error analysis (cont'd)

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Using these formulas,

$$f''(x) = \frac{1}{h^2} \left( f(x+h) - 2f(x) + f(x-h) \right) - \frac{h^2}{12} f^{(4)}(\xi),$$

$$g'(t) = \frac{1}{k} \left( g(t+k) - g(t) \right) - \frac{k}{2} g''(\tau),$$

with  $sh^2 = k$  and  $u_{xx} = u_t$ , we obtain

$$\begin{aligned} e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) - s(h^2 u_{xx}(x_i, t_j) + \frac{h^4}{12} u_{xxxx}(\xi_i, t_j)) \\ &\quad + (ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, \tau_j)), \\ \Rightarrow e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) \\ &\quad - kh^2 \left( \frac{1}{12} u_{xxxx}(\xi_i, t_j) - \frac{s}{2} u_{tt}(x_i, \tau_j) \right). \end{aligned}$$

## Error analysis (cont'd)

- Let us confine  $(x, t)$  to the compact set

$$S = \{(x, t) : 0 \leq x \leq 1, 0 \leq t \leq T\}.$$

$$\text{Put } M = \frac{1}{12} \max_S |u_{xxxx}(x, t)| + \frac{1}{2} \max_S |u_{tt}(x, t)|,$$

$$E_j = (e_{1,j}, e_{2,j}, \dots, e_{n,j})^\top, \|E_j\|_\infty = \max_{1 \leq i \leq n} |e_{ij}|.$$

- We assume  $1 - 2s \geq 0$ :

$$\begin{aligned} |e_{i,j+1}| &\leq s|e_{i-1,j}| + (1 - 2s)|e_{ij}| + s|e_{i+1,j}| + kh^2M \\ &\leq s\|E_j\|_\infty + (1 - 2s)\|E_j\|_\infty + s\|E_j\|_\infty + kh^2M \\ &\leq \|E_j\|_\infty + kh^2M. \end{aligned}$$

Hence,

$$\begin{aligned} \|E_{j+1}\|_\infty &\leq \|E_j\|_\infty + kh^2M \leq \|E_{j-1}\|_\infty + 2kh^2M \\ &\leq \dots \leq \|E_0\|_\infty + (j+1)kh^2M. \end{aligned}$$

$$\implies \|E_j\|_\infty \leq jkh^2M \implies \|E_j\|_\infty \leq Th^2M = O(h^2).$$

## Numerical differentiation

Assume that  $u \in C^4[a, b]$  and  $a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b$  is a uniform partition of  $[a, b]$ . Then  $h_j = h = \frac{b-a}{M+1}$  for  $j = 1, 2, \cdots, M+1$ . For  $i = 1, 2, \cdots, M$ , we have

$$u(x_i + h) = u(x_i) + u'(x_i)h + \frac{1}{2}u''(x_i)h^2 + \frac{1}{6}u^{(3)}(x_i)h^3 + \frac{1}{24}u^{(4)}(\xi_{i1})h^4,$$

$$u(x_i - h) = u(x_i) - u'(x_i)h + \frac{1}{2}u''(x_i)h^2 - \frac{1}{6}u^{(3)}(x_i)h^3 + \frac{1}{24}u^{(4)}(\xi_{i2})h^4,$$

for some  $\xi_{i1} \in (x_i, x_i + h)$  and  $\xi_{i2} \in (x_i - h, x_i)$ . Therefore, we have

$$u(x_i + h) + u(x_i - h) = 2u(x_i) + u''(x_i)h^2 + \frac{1}{24}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}h^4.$$

$$\therefore u''(x_i) =$$

$$\frac{1}{h^2}\{u(x_i + h) - 2u(x_i) + u(x_i - h)\} - \frac{1}{24}h^2\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$$

$$\therefore u \in C^4[a, b], \frac{1}{2}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\} \text{ between } u^{(4)}(\xi_{i1}) \text{ \& } u^{(4)}(\xi_{i2}).$$

$\therefore$  By IVT,  $\exists \xi_i$  between  $\xi_{i1}$  and  $\xi_{i2}$  ( $\Rightarrow \xi_i \in (x_i - h, x_i + h)$ ) such that

$$u^{(4)}(\xi_i) = \frac{1}{2}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$$

$$\therefore u''(x_i) = \frac{1}{h^2}\{u(x_i + h) - 2u(x_i) + u(x_i - h)\} - \frac{1}{12}h^2u^{(4)}(\xi_i),$$

for some  $\xi_i \in (x_i - h, x_i + h)$ . (2nd-order approximation)



## FDM for a two-point boundary value problem

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- Consider the 1-D two-point BVP:

$$\begin{cases} -u''(x) &= f(x) \quad x \in (0,1), \\ u(0) &= u(1) = 0. \end{cases}$$

- The interval  $[0,1]$  is discretized uniformly by taking the  $n+2$  points,  $x_i = ih$ , for  $i = 0, 1, \dots, n+1$ , where  $h = 1/(n+1)$ .
- Let  $v_i \approx u(x_i)$ ,  $i = 1, 2, \dots, n$ , and  $v_0 := u(x_0) = 0$ ,  $v_{n+1} := u(x_{n+1}) = 0$  are known due to the Dirichlet BC.
- If the centered difference approximation is used for  $u''$ , the above equation can be expressed as

$$-\left( \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} \right) = f_i, \quad i = 1, 2, \dots, n,$$

where  $f_i := f(x_i)$ .

## Resulting linear system

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The linear system obtained is of the form

$$AV = F,$$

where

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

$$V = (v_1, v_2, \dots, v_n)^\top \quad \text{and} \quad F = (h^2 f_1, h^2 f_2, \dots, h^2 f_n)^\top.$$

## Eigen properties of $A$

- The matrix  $A$  has  $n$  eigenvalues, and since  $A$  is symmetric, all eigenvalues must be real.
- Note that the eigenvalues of  $A$  are given by

$$\lambda_j = 2 - 2 \cos(j\theta) > 0, j = 1, 2, \dots, n,$$

and the eigenvector associated with each  $\lambda_j$  is given by

$$V_j = (\sin(j\theta), \sin(2j\theta), \dots, \sin(nj\theta))^T,$$

where  $\theta = \frac{\pi}{n+1}$ .

- $\lambda_{\max} = 2 - 2 \cos(\frac{n\pi}{n+1})$  and  $\lambda_{\min} = 2 - 2 \cos(\frac{\pi}{n+1})$ .
- What is the condition number of  $A$ ?  
( $\sin x = x - x^3/(3!) + x^5/(5!) \dots$ )

$$\kappa(A) = \frac{\sin^2 \frac{n\pi}{2(n+1)}}{\sin^2 \frac{\pi}{2(n+1)}} \approx \frac{\sin(\pi/2)}{(\frac{\pi}{2(n+1)})^2} \approx O(n^2) \approx O\left(\frac{1}{h^2}\right).$$

## FDM for a 2-D boundary value problem

- Consider Poisson's problem,

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = f & \text{in } \Omega := (0,1) \times (0,1), \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

- Define the mesh size  $h = \frac{1}{n+1}$ , the collection of mesh points  $(x_{1i}, x_{2j}) = (ih, jh)$ , the approximate solution at the mesh points  $v_{ij} \approx u(x_{1j}, x_{2j})$ ,  $i, j = 0, 1, \dots, n+1$ .

**Note:** There are  $n^2$  interior points  $\approx \frac{1}{h^2}$ . (in 3D,  $\approx \frac{1}{h^3}$  number of points).

- The FD equations

$$\begin{cases} -\left(\frac{v_{i-1j} - 2v_{ij} + v_{i+1j}}{h^2} + \frac{v_{ij-1} - 2v_{ij} + v_{ij+1}}{h^2}\right) = f_{ij}, \\ v_{0j} = v_{n+1j} = v_{i0} = v_{in+1} = 0. \end{cases}$$

## For example $n = 3$ : natural ordering

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- We order the unknown quantities in the natural ordering

$$V = (v_{11}, v_{21}, v_{31}, v_{12}, v_{22}, v_{n2}, v_{13}, v_{23}, v_{33})^\top.$$

- Then the corresponding linear system can be written as (see Text, page 631)

$$AV = \begin{bmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{bmatrix} V = F \quad \text{with} \quad B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

- block-tridiag matrix; symmetric  $a_{ij} = a_{ji}$ ; sparse, number of nonzeros per row  $\approx 5$  (independent of the mesh size  $h$ ) number of nonzeros  $\approx 5n$  (linear in  $n$ ).

## References

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- K. W. Morton and D. F. Mayers, *Numerical Solution of Partial Differential Equations*, Cambridge University Press, 1994.
- K. W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman & Hall, 1996.