MA2007B: LINEAR ALGEBRA I

Final Exam/January 14, 2021

Please show all your work clearly for full credit! total 110 points

(1) (10 pts) Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix}$. Find the complete solution to the linear system.

Solution:

$$[\boldsymbol{A}|\boldsymbol{b}] = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} := [\boldsymbol{R}|\boldsymbol{d}]$$

First and third are pivot columns, second and fourth are free columns.

Note that $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Rx} = \boldsymbol{0}$

Let $x_2 = s$ and $x_4 = t$. Then $x_3 = -4x_4 = -4t$ and $x_1 = -3x_2 - 2x_4 = -3s - 2t$.

$\therefore$ The solutions to $\boldsymbol{Ax} = \boldsymbol{0}$ are

$$\boldsymbol{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \forall\, s, t \in \mathbb{R}.$$

Let the free variables $x_2 = 0 = x_4$. A particular solution to $\boldsymbol{Ax} = \boldsymbol{b} (\Leftrightarrow \boldsymbol{Rx} = \boldsymbol{d})$ is

$$\boldsymbol{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \end{bmatrix}.$$

Therefore, the complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \forall\, s, t \in \mathbb{R}.$$

(2) (10 pts) Let $\boldsymbol{V}$ be a subspace of $\mathbb{R}^n$ and $\boldsymbol{V}^\perp := \{\boldsymbol{x} \in \mathbb{R}^n |\ \boldsymbol{x} \cdot \boldsymbol{v} = 0,\ \forall \boldsymbol{v} \in \boldsymbol{V}\}$ be the orthogonal complement of $\boldsymbol{V}$.

(a) Show that $\boldsymbol{V}^\perp$ is also a subspace of $\mathbb{R}^n$.

(b) Show that $\boldsymbol{V} \cap \boldsymbol{V}^\perp = \{\boldsymbol{0}\}$.

Proof:

(a) Claim: $\boldsymbol{V}^\perp$ is a subspace of $\mathbb{R}^n$

(i) Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}^\perp$. Then $\boldsymbol{x} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{y} \cdot \boldsymbol{v} = 0$, $\forall\, \boldsymbol{v} \in \boldsymbol{V}$.

$\therefore\ (\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{v} = \boldsymbol{x} \cdot \boldsymbol{v} + \boldsymbol{y} \cdot \boldsymbol{v} = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}$

$\therefore\ \boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{V}^\perp$

(ii) Let $\boldsymbol{x} \in \boldsymbol{V}^\perp$ and $\alpha \in \mathbb{R}$. Then $\boldsymbol{x} \cdot \boldsymbol{v} = 0$, $\forall\, \boldsymbol{v} \in \boldsymbol{V}$.

$\therefore\ (\alpha \boldsymbol{x}) \cdot \boldsymbol{v} = \alpha(\boldsymbol{x} \cdot \boldsymbol{v}) = 0,\ \forall\, \boldsymbol{v} \in \boldsymbol{V}$

$\therefore\ \alpha \boldsymbol{x} \in \boldsymbol{V}^\perp$

(b) Claim: $\boldsymbol{V} \cap \boldsymbol{V}^{\perp} = \{\mathbf{0}\}$
$\because$ $\boldsymbol{V}$ and $\boldsymbol{V}^{\perp}$ are subspaces of $\mathbb{R}^n$
$\therefore$ $\mathbf{0} \in \boldsymbol{V}$ and $\mathbf{0} \in \boldsymbol{V}^{\perp}$
$\therefore$ $\mathbf{0} \in \boldsymbol{V} \cap \boldsymbol{V}^{\perp}$
Suppose that $\exists\, \boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{x} \in \boldsymbol{V} \cap \boldsymbol{V}^{\perp}$.
Then $\boldsymbol{x} \in \boldsymbol{V}$ and $\boldsymbol{x} \in \boldsymbol{V}^{\perp}$
$\therefore$ $\boldsymbol{x} \cdot \boldsymbol{x} = 0 \Rightarrow \|\boldsymbol{x}\|^2 = 0 \Rightarrow \|\boldsymbol{x}\| = 0 \Rightarrow \boldsymbol{x} = \mathbf{0}$, a contradiction!
$\therefore$ $\boldsymbol{V} \cap \boldsymbol{V}^{\perp} = \{\mathbf{0}\}$

(3) (10 pts) Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$.

(a) Show that $\boldsymbol{A}^\top \boldsymbol{A}$ has the same nullspace as $\boldsymbol{A}$, i.e., $N(\boldsymbol{A}^\top \boldsymbol{A}) = N(\boldsymbol{A})$.
(b) Show that if $\boldsymbol{B}$ has full row rank, then $\boldsymbol{B}\boldsymbol{B}^\top$ is invertible.

Proof:

(a) Note that $N(\boldsymbol{A}) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\}$ and $N(\boldsymbol{A}^\top \boldsymbol{A}) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\}$.
(i) If $\boldsymbol{x} \in N(\boldsymbol{A})$, then $\boldsymbol{A}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^\top \mathbf{0} = \mathbf{0} \implies \boldsymbol{x} \in N(\boldsymbol{A}^\top \boldsymbol{A})$
(ii) If $\boldsymbol{x} \in N(\boldsymbol{A}^\top \boldsymbol{A})$, then $\boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x}^\top \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\top \mathbf{0} = 0$
$\implies (\boldsymbol{A}\boldsymbol{x})^\top \boldsymbol{A}\boldsymbol{x} = 0 \implies \|\boldsymbol{A}\boldsymbol{x}\|^2 = 0 \implies \boldsymbol{A}\boldsymbol{x} = \mathbf{0} \implies \boldsymbol{x} \in N(\boldsymbol{A})$
By (i) and (ii), $N(\boldsymbol{A}^\top \boldsymbol{A}) = N(\boldsymbol{A})$.

(b) Let $\boldsymbol{A} := \boldsymbol{B}^\top \in \mathbb{R}^{m \times n}$. Then $\boldsymbol{A}$ has full column rank.
$\therefore$ The columns of $\boldsymbol{A}$ are linearly independent
$\therefore$ $N(\boldsymbol{A}) = \{\mathbf{0}\}$
$\therefore$ By part (a), $N(\boldsymbol{A}^\top \boldsymbol{A}) = N(\boldsymbol{A}) = \{\mathbf{0}\}$
$\therefore$ The columns of the $n \times n$ square matrix $\boldsymbol{A}^\top \boldsymbol{A}$ are linearly independent
$\therefore$ $\boldsymbol{A}^\top \boldsymbol{A}$ is invertible
$\because$ $\boldsymbol{B}\boldsymbol{B}^\top = \boldsymbol{A}^\top \boldsymbol{A}$
$\therefore$ $\boldsymbol{B}\boldsymbol{B}^\top$ is invertible

(4) (10 pts) Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{R}^n$. Show that there exist $\boldsymbol{x}_r \in C(\boldsymbol{A}^\top)$ (the row space of $\boldsymbol{A}$) and $\boldsymbol{x}_n \in N(\boldsymbol{A})$ (the nullspace of $\boldsymbol{A}$) such that $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$ and the representation is unique.

Proof:

(i) Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ be a basis for $C(\boldsymbol{A}^\top) \subseteq \mathbb{R}^n$
and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n-r}\}$ be a basis for $N(\boldsymbol{A}) \subseteq \mathbb{R}^n$.
Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r, \boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n-r}\}$ is a basis for $\mathbb{R}^n$.
$\because$ $\boldsymbol{x} \in \mathbb{R}^n$
$\therefore$ $\exists!\, \alpha_1, \alpha_2, \cdots, \alpha_r, \beta_1, \beta_2, \cdots, \beta_{n-r}$ such that
$$\boldsymbol{x} = \underbrace{\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_r \boldsymbol{v}_r}_{:=\boldsymbol{x}_r} + \underbrace{\beta_1 \boldsymbol{w}_1 + \beta_2 \boldsymbol{w}_2 + \cdots + \beta_{n-r} \boldsymbol{w}_{n-r}}_{:=\boldsymbol{x}_n}$$
$\therefore$ $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r \in C(\boldsymbol{A}^\top)$ and $\boldsymbol{x}_n \in N(\boldsymbol{A})$

(ii) Suppose that $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, $\boldsymbol{x}_r \in C(\boldsymbol{A}^\top)$ and $\boldsymbol{x}_n \in N(\boldsymbol{A})$ and
$\boldsymbol{x} = \boldsymbol{x}_r' + \boldsymbol{x}_n'$, where $\boldsymbol{x}_r' \in C(\boldsymbol{A}^\top)$ and $\boldsymbol{x}_n' \in N(\boldsymbol{A})$
Then $\mathbf{0} = \boldsymbol{x} - \boldsymbol{x} = (\boldsymbol{x}_r - \boldsymbol{x}_r') + (\boldsymbol{x}_n - \boldsymbol{x}_n')$.
$\therefore$ $\boldsymbol{x}_r - \boldsymbol{x}_r' = -(\boldsymbol{x}_n - \boldsymbol{x}_n')$
$\because$ $\boldsymbol{x}_r - \boldsymbol{x}_r' \in C(\boldsymbol{A}^\top)$, $-(\boldsymbol{x}_n - \boldsymbol{x}_n') \in N(\boldsymbol{A})$, and $C(\boldsymbol{A}^\top) \cap N(\boldsymbol{A}) = \{\mathbf{0}\}$
$\therefore$ $\boldsymbol{x}_r - \boldsymbol{x}_r' = \mathbf{0}$ and $-(\boldsymbol{x}_n - \boldsymbol{x}_n') = \mathbf{0} \implies \boldsymbol{x}_r = \boldsymbol{x}_r'$ and $\boldsymbol{x}_n = \boldsymbol{x}_n'$
$\therefore$ The representation $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$ is unique

(5) (10 pts) Let $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n] \in \mathbb{R}^{m\times n}$, where $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n \in \mathbb{R}^m$ are linearly independent. Let $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{b} \notin C(\boldsymbol{A})$, where $C(\boldsymbol{A})$ denotes the column space of $\boldsymbol{A}$.

(a) Show that the orthogonal projection of $\boldsymbol{b}$ onto the column space $C(\boldsymbol{A})$ is $\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}}$, where $\widehat{\boldsymbol{x}}$ is the solution of the normal equation $\boldsymbol{A}^\top\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^\top\boldsymbol{b}$, and explain why the normal equation has a unique solution $\widehat{\boldsymbol{x}}$.

(b) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Find the orthogonal projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space $C(\boldsymbol{A})$.

Proof:

(a) Let $\boldsymbol{p}$ be the orthogonal projection of $\boldsymbol{b}$ onto the column space $C(\boldsymbol{A})$.
Then $\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}}$ for some $\widehat{\boldsymbol{x}} \in \mathbb{R}^n$.
$\because$ $\boldsymbol{b} - \boldsymbol{p} = \boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}$ is perpendicular to the column space $C(\boldsymbol{A})$
$\therefore$ $(\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) \perp \boldsymbol{a}_i, \forall\, i = 1, 2, \cdots, n$
$\therefore$ $\boldsymbol{a}_i \cdot (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = 0, \forall\, i = 1, 2, \cdots, n$
$\therefore$ $\boldsymbol{a}_i^\top(\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = 0, \forall\, i = 1, 2, \cdots, n$

$$\therefore \begin{bmatrix} \boldsymbol{a}_1^\top \\ \boldsymbol{a}_2^\top \\ \vdots \\ \boldsymbol{a}_n^\top \end{bmatrix} (\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore$ $\boldsymbol{A}^\top(\boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}) = \boldsymbol{0}$
$\therefore$ $\boldsymbol{A}^\top\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^\top\boldsymbol{b}$
$\because$ The columns of $\boldsymbol{A}$ are linearly independent
$\therefore$ The columns of $\boldsymbol{A}^\top\boldsymbol{A}$ are linearly independent (see the proof of 3(b))
$\therefore$ $\boldsymbol{A}^\top\boldsymbol{A}$ is invertible
$\therefore$ The normal equation $\boldsymbol{A}^\top\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^\top\boldsymbol{b}$ has a unique solution $\widehat{\boldsymbol{x}}$

(b) By direct calculations, we have

$$\boldsymbol{A}^\top\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix},$$

$$\boldsymbol{A}^\top\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

The normal equation is

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \implies \widehat{\boldsymbol{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Therefore, the orthogonal projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto the column space $C(\boldsymbol{A})$ is

$$\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

(6) (15 pts) Let $\boldsymbol{Q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_n] \in \mathbb{R}^{m\times n}$ be a matrix with orthonormal columns.

(a) Show that $\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_n$ are linearly independent.

(b) Show that $\|\boldsymbol{Qx}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

(c) Show that $\boldsymbol{Qx} \cdot \boldsymbol{Qy} = \boldsymbol{x} \cdot \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Proof:

(a) Let $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + \cdots + c_n\boldsymbol{q}_n = \boldsymbol{0}$.
Then $(c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + \cdots + c_n\boldsymbol{q}_n) \cdot \boldsymbol{q}_1 = \boldsymbol{0} \cdot \boldsymbol{q}_1 = 0$.
$\therefore\ c_1\boldsymbol{q}_1 \cdot \boldsymbol{q}_1 + c_2\boldsymbol{q}_2 \cdot \boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n \cdot \boldsymbol{q}_1 = 0$
$\because\ \boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_n$ are orthonormal columns of $\boldsymbol{Q}$
$\therefore\ \boldsymbol{q}_i \cdot \boldsymbol{q}_j = 0$ if $i \neq j$
$\therefore\ c_1\boldsymbol{q}_1 \cdot \boldsymbol{q}_1 = 0$
$\because\ \boldsymbol{q}_1 \cdot \boldsymbol{q}_1 = \|\boldsymbol{q}_1\|^2 > 0 \quad (\because\ \boldsymbol{q}_1 \neq \boldsymbol{0})$
$\therefore\ c_1 = 0$
Similarly, we can prove $c_2 = 0, \cdots, c_n = 0$.
$\therefore\ \boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_n$ are linearly independent

(b) $\|\boldsymbol{Qx}\|^2 = \boldsymbol{Qx} \cdot \boldsymbol{Qx} = (\boldsymbol{Qx})^\top \boldsymbol{Qx} = \boldsymbol{x}^\top \boldsymbol{Q}^\top \boldsymbol{Qx} = \boldsymbol{x}^\top \boldsymbol{Ix} = \boldsymbol{x}^\top \boldsymbol{x} = \|\boldsymbol{x}\|^2$
$\Longrightarrow \|\boldsymbol{Qx}\| = \|\boldsymbol{x}\|, \forall\, \boldsymbol{x} \in \mathbb{R}^n$

(c) $\boldsymbol{Qx} \cdot \boldsymbol{Qy} = (\boldsymbol{Qx})^\top \boldsymbol{Qy} = \boldsymbol{x}^\top \boldsymbol{Q}^\top \boldsymbol{Qy} = \boldsymbol{x}^\top \boldsymbol{Iy} = \boldsymbol{x}^\top \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}, \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$

(7) (15 pts) Let $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3] = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$.

(a) Show that the columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent.

(b) Find the orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ by the Gram-Schmidt process.

(c) Find the factorization, $\boldsymbol{A} = \boldsymbol{QR}$, by using part (b), where $\boldsymbol{Q}$ is an orthogonal matrix and $\boldsymbol{R}$ is an upper triangular matrix.

Solution:

(a) $\because\ \det\boldsymbol{A} = -\det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} = -15 \neq 0$
$\therefore\ \boldsymbol{A}$ is nonsingular
$\therefore$ the columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent

(b) By the Gram-Schmidt process, we have

$$A := \boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{q}_1 = \frac{A}{\|A\|} = \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$B = \boldsymbol{b} - \frac{A^\top \boldsymbol{b}}{A^\top A}A = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \Longrightarrow \boldsymbol{q}_2 = \frac{B}{\|B\|} = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$C = \boldsymbol{c} - \frac{A^\top \boldsymbol{c}}{A^\top A}A - \frac{B^\top \boldsymbol{c}}{B^\top B}B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{18}{9}\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \boldsymbol{q}_3 = \frac{C}{\|C\|} = \frac{1}{5}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

(c) From part (b), we obtain

$$A = QR = [q_1 \quad q_2 \quad q_3] \begin{bmatrix} q_1^\top a & q_1^\top b & q_1^\top c \\ 0 & q_2^\top b & q_2^\top c \\ 0 & 0 & q_3^\top c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

(8) (15 pts) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Assume that through the Gaussian elimination process, we obtain $\boldsymbol{PA} = \boldsymbol{LU}$, where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is the lower triangular matrix, and $\boldsymbol{U}$ is the upper triangular matrix. Show that $\det(\boldsymbol{A}^\top) = \det(\boldsymbol{A})$.

Proof:

$\because\ \boldsymbol{PA} = \boldsymbol{LU}$

$\therefore\ (\boldsymbol{PA})^\top = (\boldsymbol{LU})^\top \implies \boldsymbol{A}^\top \boldsymbol{P}^\top = \boldsymbol{U}^\top \boldsymbol{L}^\top$

$\implies \det(\boldsymbol{P})\det(\boldsymbol{A}) = \det(\boldsymbol{L})\det(\boldsymbol{U}) \quad \text{and} \quad \det(\boldsymbol{A}^\top)\det(\boldsymbol{P}^\top) = \det(\boldsymbol{U}^\top)\det(\boldsymbol{L}^\top)$

$\because\ \det(\boldsymbol{U}) = \det(\boldsymbol{U}^\top) \quad (\because$ have the same diagonal),

$\det(\boldsymbol{L}) = \det(\boldsymbol{L}^\top) = 1 \quad (\because$ both have 1's on the diagonal),

$\therefore\ \det(\boldsymbol{P})\det(\boldsymbol{A}) = \det(\boldsymbol{A}^\top)\det(\boldsymbol{P}^\top)$

$\because\ \boldsymbol{P}^\top\boldsymbol{P} = \boldsymbol{I} \implies \det(\boldsymbol{P}^\top\boldsymbol{P}) = \det(\boldsymbol{P}^\top)\det(\boldsymbol{P}) = 1$

$\because\ \boldsymbol{P}$ and $\boldsymbol{P}^\top$ are permutation matrices

$\therefore\ \det(\boldsymbol{P}) = 1 = \det(\boldsymbol{P}^\top) \quad \text{or} \quad \det(\boldsymbol{P}) = -1 = \det(\boldsymbol{P}^\top)$

$\therefore\ \det(\boldsymbol{P}) = \det(\boldsymbol{P}^\top)$

$\therefore\ \det(\boldsymbol{A}) = \det(\boldsymbol{A}^\top)$

(9) (15 pts) Compute the determinants of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Are their rows linearly independent? Please give your reasons.

Solution:

$\det \boldsymbol{A} = 1 + 12 + 18 - 9 - 4 - 6 = 12 \neq 0 \implies \boldsymbol{A}$ is nonsingular

$\implies \cdots \implies$ rows of $\boldsymbol{A}$ are linearly independent

$\det \boldsymbol{B} = 28 + 40 + 72 - 60 - 24 - 56 = 0 \implies \boldsymbol{B}$ is singular

$\implies \cdots \implies$ rows of $\boldsymbol{B}$ are linearly dependent (in fact, row3 - row1 = row2)

$\det \boldsymbol{C} = 0 + 0 + 0 - 1 - 0 - 0 = -1 \neq 0 \implies \boldsymbol{C}$ is nonsingular

$\implies \cdots \implies$ rows of $\boldsymbol{C}$ are linearly independent