MA2007B: LINEAR ALGEBRA I

Midterm1/October 29, 2020

Please show all your work clearly for full credit!

(1) In this problem, we consider the vectors in $\mathbb{R}^2$.

(1a) (10 pts) Let $\boldsymbol{u}$ and $\boldsymbol{U}$ be two unit vectors in $\mathbb{R}^2$ and θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{U}$. Show that $\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta$.

(1b) (5 pts) Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two nonzero vectors in $\mathbb{R}^2$. Show that $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(1c) (5 pts) Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vectors in $\mathbb{R}^2$. Show that $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

Solution:

(1a) Let α be the angle between vector $\boldsymbol{U}$ and the x-axis, then $\boldsymbol{U} = [\cos\alpha, \sin\alpha]^\top$.
Let β be the angle between vector $\boldsymbol{u}$ and the x-axis, then $\boldsymbol{u} = [\cos\beta, \sin\beta]^\top$.
Assume that $\beta > \alpha$. Then $\theta := \beta - \alpha$ is the angle between $\boldsymbol{u}$ and $\boldsymbol{U}$.
$\therefore\ \boldsymbol{u} \cdot \boldsymbol{U} = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \cos(\beta-\alpha) = \cos\theta$

(1b) $\because\ \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ and $\dfrac{\boldsymbol{w}}{\|\boldsymbol{w}\|}$ are two unit vectors
By (1a), we have $\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \dfrac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \cos\theta$, where θ is is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.
$\therefore\ \boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$

(1c) Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vectors in $\mathbb{R}^2$. Then

$$\begin{aligned}
\|\boldsymbol{v}+\boldsymbol{w}\|^2 &= (\boldsymbol{v}+\boldsymbol{w}) \cdot (\boldsymbol{v}+\boldsymbol{w}) \\
&= \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \\
&= \|\boldsymbol{v}\|^2 + 2\boldsymbol{v}\cdot\boldsymbol{w} + \|\boldsymbol{w}\|^2 \\
&= \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2 \\
&\le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2 \\
&= (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2.
\end{aligned}$$

$\therefore\ \|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$

(2) A norm on $\mathbb{R}^n$ is a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies three conditions.

(2a) (5 pts) What are the three conditions for a norm?

(2b) (10 pts) Define a function $f(\cdot) := \|\cdot\|_\infty$ on $\mathbb{R}^n$ by

$$f(\boldsymbol{v}) := \|\boldsymbol{v}\|_\infty := \max\{|v_1|, |v_2|, \cdots, |v_n|\}, \quad \forall\ \boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n.$$

Show that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$.

(2c) (5 pts) Please draw the three unit circles in $\mathbb{R}^2$:

$$C_1 := \{\boldsymbol{v} \in \mathbb{R}^2 : \ \|\boldsymbol{v}\|_1 = 1\}, \quad C_2 := \{\boldsymbol{v} \in \mathbb{R}^2 : \ \|\boldsymbol{v}\| = 1\}, \quad C_3 := \{\boldsymbol{v} \in \mathbb{R}^2 : \ \|\boldsymbol{v}\|_\infty = 1\}.$$

Solution:

(2a) The three conditions are

(i) $f(\boldsymbol{v}) \ge 0, \forall\ \boldsymbol{v} \in \mathbb{R}^n$ and $f(\boldsymbol{v}) = 0 \ \Leftrightarrow\ \boldsymbol{v} = \boldsymbol{0}$.

(ii) $f(\alpha\boldsymbol{v}) = |\alpha| f(\boldsymbol{v}), \forall\ \boldsymbol{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

(iii) $f(\boldsymbol{v}+\boldsymbol{w}) \le f(\boldsymbol{v}) + f(\boldsymbol{w}), \forall\, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. (called the triangle inequality)

(2b) (i) $\forall\, \boldsymbol{v} \in \mathbb{R}^n$, we have $\|\boldsymbol{v}\|_\infty = \max\{|v_1|, |v_2|, \cdots, |v_n|\} \ge 0$, since $|v_i| \ge 0\ \forall\, i$.
$\|\boldsymbol{v}\|_\infty = \max\{|v_1|, |v_2|, \cdots, |v_n|\} = 0 \Longleftrightarrow |v_i| = 0, 1 \le i \le n, \Longleftrightarrow \boldsymbol{v} = \boldsymbol{0}$

(ii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then $\alpha\boldsymbol{v} = (\alpha v_1, \alpha v_2, \cdots, \alpha v_n)$ and

$$\begin{aligned} \|\alpha\boldsymbol{v}\|_\infty &= \max\{|\alpha v_1|, |\alpha v_2|, \cdots, |\alpha v_n|\} \\ &= \max\{|\alpha||v_1|, |\alpha||v_2|, \cdots, |\alpha||v_n|\} \\ &= |\alpha| \max\{|v_1|, |v_2|, \cdots, |v_n|\} = |\alpha|\|\boldsymbol{v}\|_\infty. \end{aligned}$$

(iii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n), \boldsymbol{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} \|\boldsymbol{v}+\boldsymbol{w}\|_\infty &= \|(v_1, v_2, \cdots, v_n) + (w_1, w_2, \cdots, w_n)\|_\infty \\ &= \|(v_1+w_1, v_2+w_2, \cdots, v_n+w_n)\|_\infty \\ &= \max\{|v_1+w_1|, |v_2+w_2|, \cdots, |v_n+w_n|\} \\ &\le \max\{|v_1|+|w_1|, |v_2|+|w_2|, \cdots, |v_n|+|w_n|\} \quad (\text{since } |v_i+w_i| \le |v_i|+|w_i|\ \forall\, i) \\ &\le \max\{|v_1|, |v_2|, \cdots, |v_n|\} + \max\{|w_1|, |w_2|, \cdots, |w_n|\} \\ &= \|\boldsymbol{v}\|_\infty + \|\boldsymbol{w}\|_\infty. \quad \square \end{aligned}$$

(2c) C_1 is a diamond; C_2 is a usual circle; C_3 is a square!
(We omit the graphs)

(3) Consider the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where matrix $\boldsymbol{A}$ is the 4×4 Pascal matrix,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

(3a) (5 pts) Apply the Gaussian elimination method to factor $\boldsymbol{A}$ into $\boldsymbol{LU}$, i.e., $\boldsymbol{A} = \boldsymbol{LU}$, where $\boldsymbol{L}$ is a lower triangular matrix and $\boldsymbol{U}$ is an upper triangular matrix.

(3b) (5 pts) What are the pivots p_i and the multipliers ℓ_{ij} in (3a)?

(3c) (10 pts) Let $\boldsymbol{b} = [4\ \ 10\ \ 20\ \ 35]^\top$. Find the solution $\boldsymbol{x} = [x_1\ \ x_2\ \ x_3\ \ x_4]^\top$ of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ by solving two triangular systems, one with the lower triangular matrix $\boldsymbol{L}$ and the other with the upper triangular matrix $\boldsymbol{U}$, both derived in problem (3a).

Solution:

(3a)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \xrightarrow{\ell_{21}=\ell_{31}=\ell_{41}=\frac{1}{1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} \xrightarrow{\ell_{32}=\frac{2}{1},\ \ell_{42}=\frac{3}{1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix}$$

$$\xrightarrow{\ell_{43}=\frac{3}{1}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} := \boldsymbol{U}.$$

Then we have

$$\boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(3b) Pivots: $p_1 = p_2 = p_3 = p_4 = 1$;
Multipliers: $\ell_{21} = \ell_{31} = \ell_{41} = 1$, $\ell_{32} = 2$, $\ell_{42} = 3$, $\ell_{43} = 3$.

(3c) $\boldsymbol{Ax} = \boldsymbol{b} = [4\ \ 10\ \ 20\ \ 35]^\top \Longrightarrow \boldsymbol{LUx} = [4\ \ 10\ \ 20\ \ 35]^\top$
Let $\boldsymbol{Ux} = \boldsymbol{c}$. Then we first solve $\boldsymbol{Lc} = [4\ \ 10\ \ 20\ \ 35]^\top$. That is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 20 \\ 35 \end{bmatrix}.$$

Therefore, $c_1 = 4 \Longrightarrow c_2 = 10 - c_1 = 6 \Longrightarrow c_3 = 20 - c_1 - 2c_2 = 4$
$\Longrightarrow c_4 = 35 - c_1 - 3c_2 - 3c_3 = 1$.
We then solve $\boldsymbol{Ux} = \boldsymbol{c}$, that is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 1 \end{bmatrix}.$$

Therefore, $x_4 = 1 \Longrightarrow x_3 = 4 - 3x_4 = 1 \Longrightarrow x_2 = 6 - 2x_3 - 3x_4 = 1$
$\Longrightarrow x_1 = 4 - x_2 - x_3 - x_4 = 1$.

$\therefore$ the solution is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(4) (10 pts) Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be three $n \times n$ matrices. Prove that if $\boldsymbol{ABC}$ is invertible, then $\boldsymbol{B}$ is invertible.

Proof:

$\because$ $\boldsymbol{ABC}$ is invertible

$\therefore$ $\exists$ $n \times n$ matrix $\boldsymbol{D}$ such that $\boldsymbol{D}(\boldsymbol{ABC}) = (\boldsymbol{ABC})\boldsymbol{D} = \boldsymbol{I}$

$\because$ $\boldsymbol{A}(\boldsymbol{BCD}) = (\boldsymbol{ABC})\boldsymbol{D} = \boldsymbol{I}$

$\therefore$ $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \boldsymbol{BCD}$

$\because$ $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

$\therefore$ $\boldsymbol{A}^{-1}$ is invertible

$\because$ $\boldsymbol{ABC}$ and $\boldsymbol{A}^{-1}$ are invertible

$\therefore$ $\boldsymbol{A}^{-1}(\boldsymbol{ABC}) = \boldsymbol{IBC} = \boldsymbol{BC}$ is invertible

$\because$ $\boldsymbol{BC}$ is invertible

$\therefore$ $\exists$ $n \times n$ matrix $\boldsymbol{E}$ such that $\boldsymbol{E}(\boldsymbol{BC}) = (\boldsymbol{BC})\boldsymbol{E} = \boldsymbol{I}$

$\because$ $\boldsymbol{B}(\boldsymbol{CE}) = (\boldsymbol{BC})\boldsymbol{E} = \boldsymbol{I}$

$\therefore$ $\boldsymbol{B}$ is invertible

(5) (10 pts) Use the permutation and elimination to find a factorization of $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \boldsymbol{BU},$$

where $\boldsymbol{U}$ is the upper triangular matrix

$$\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Please express matrix $\boldsymbol{B}$ explicitly as a product of permutation and elimination matrices.

Solution:

Let $\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a permutation matrix. Then we have

$$\boldsymbol{PA} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{\ell_{31}=\frac{2}{1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{\ell_{32}=\frac{3}{1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} := \boldsymbol{U}$$

Therefore, we have $\boldsymbol{E}_{32}\boldsymbol{E}_{31}\boldsymbol{PA} = \boldsymbol{U}$, which implies that

$$\boldsymbol{PA} = \boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}}_{:=\boldsymbol{L}} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{\boldsymbol{U}} = \boldsymbol{LU}$$

$\because$ $\boldsymbol{P}$ is a permutation matrix and $\boldsymbol{PP}^\top = \boldsymbol{I}$

$$\therefore\ \boldsymbol{P}^{-1} = \boldsymbol{P}^\top = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore\ \boldsymbol{A} = \boldsymbol{P}^\top\boldsymbol{E}_{31}^{-1}\boldsymbol{E}_{32}^{-1}\boldsymbol{U} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} := \boldsymbol{BU},$$

where $\boldsymbol{B} := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$

(6) (10 pts) Find the symmetric factorization $\boldsymbol{A} = \boldsymbol{LDL}^\top$ for the matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\ell_{21}=\frac{-1}{2}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{\ell_{32}=\frac{-2}{3}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow{\ell_{43}=\frac{-3}{4}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} := \boldsymbol{U}.$$

We have

$$\boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix},$$

$$
U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}}_{:=D} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{L^\top}.
$$

Therefore,

$$
A = LDL^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

(7) (10 pts) Is the 5×5 matrix A invertible? Give your complete reason.

$$
A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.
$$

Solution: No! A is not invertible. $\because\ x = [1,1,1,1,1]^\top \neq \mathbf{0}$ but $Ax = \mathbf{0}$.

Claim: If $\exists\ x \neq \mathbf{0}$ such that $Ax = \mathbf{0}$, then A is not invertible.

Suppose that A is invertible.

Then $\exists\ A^{-1} \in \mathbb{R}^{n\times n}$ such that $AA^{-1} = A^{-1}A = I$.

$\because\ \exists\ x \neq \mathbf{0}$ such that $Ax = \mathbf{0}$

$\therefore\ A^{-1}Ax = A^{-1}\mathbf{0}$

$\therefore\ Ix = \mathbf{0}$

$\therefore\ x = \mathbf{0}$. This is a contradiction!

$\therefore\ A$ is not invertible