MA2007B: LINEAR ALGEBRA I

Midterm 2/December 10, 2020

Please show all your work clearly for full credit! total 110 points

(1) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]$. Assume that $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are a basis for $\mathbb{R}^n$.

(a) (5 pts) Show that for any $\boldsymbol{v} \in \mathbb{R}^n$ there is one and only one way to write $\boldsymbol{v}$ as a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$.

(b) (10 pts) Show that $\boldsymbol{A}$ is invertible.

Proof:

(a) Suppose that $\boldsymbol{v} = \alpha_1\boldsymbol{a}_1 + \alpha_2\boldsymbol{a}_2 + \cdots + \alpha_n\boldsymbol{a}_n$ and $\boldsymbol{v} = \beta_1\boldsymbol{a}_1 + \beta_2\boldsymbol{a}_2 + \cdots + \beta_n\boldsymbol{a}_n$.
Then we have $\boldsymbol{0} = \boldsymbol{v} - \boldsymbol{v} = (\alpha_1 - \beta_1)\boldsymbol{a}_1 + (\alpha_2 - \beta_2)\boldsymbol{a}_2 + \cdots + (\alpha_n - \beta_n)\boldsymbol{a}_n$.
$\because$ $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are a basis for $\mathbb{R}^n$
$\therefore$ $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are linearly independent
$\therefore$ $\alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0, \cdots, \alpha_n - \beta_n = 0$
$\therefore$ $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \cdots, \alpha_n = \beta_n$
That is, there is one and only one way to write $\boldsymbol{v}$ as a linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$.

(b) $\because$ $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are a basis for $\mathbb{R}^n$
$\therefore$ for $i = 1, 2, \cdots, n$, $\exists\,!\ \boldsymbol{b}_i \in \mathbb{R}^n$ such that $[\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]\boldsymbol{b}_i = \boldsymbol{e}_i$ (by (1a))
Define $\boldsymbol{B} := [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n]$. Then
$\boldsymbol{AB} = [\boldsymbol{Ab}_1, \boldsymbol{Ab}_2, \cdots, \boldsymbol{Ab}_n] = [\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n] = \boldsymbol{I}$.
$\therefore$ $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$

(2) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and let r be the rank of $\boldsymbol{A}$.

(a) (10 pts) Show that the columns of $\boldsymbol{A}$ are linearly independent if and only if $r = n$.

(b) (5 pts) Assume that $m = 5$ and $n = 7$. Do the columns of matrix $\boldsymbol{A}$ be linearly independent or linearly dependent?

Proof:

(a) ($\Rightarrow$): Obviously $r \le n$. Suppose that $r < n$. Then there are free columns of $\boldsymbol{R} := rref(\boldsymbol{A})$.
$\therefore$ there are free variables of $\boldsymbol{Rx} = \boldsymbol{0}$
$\therefore$ $\exists\ \boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{Rx} = \boldsymbol{0}$
$\because$ $\boldsymbol{Rx} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{0}$
$\therefore$ $\exists\ \boldsymbol{x} \neq \boldsymbol{0}$ such that $\boldsymbol{Ax} = \boldsymbol{0}$
$\therefore$ the columns of $\boldsymbol{A}$ are linearly dependent. This is a contradiction!
$\therefore$ $r = n$
($\Leftarrow$) Let $r = n$. Then we have $\boldsymbol{R} := rref(\boldsymbol{A}) = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix}$.
$\therefore$ there are n pivots and no free variables.
$\therefore$ if $\boldsymbol{Rx} = \boldsymbol{0}$ then $\boldsymbol{Ix} = \boldsymbol{0} \Rightarrow \boldsymbol{x} = \boldsymbol{0}$
$\because$ $\boldsymbol{Rx} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{0}$
$\therefore$ if $\boldsymbol{Ax} = \boldsymbol{0}$ then $\boldsymbol{x} = \boldsymbol{0}$
$\therefore$ the columns of $\boldsymbol{A}$ are linearly independent

(b) $\because\ m = 5$ and $n = 7$

$\therefore\ r \leq 5$

$\therefore\ r < n = 7$

By (2a), the columns of $\boldsymbol{A}$ are linearly dependent.

(3) (15 pts) Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Find the complete solution to the linear system.

Solution:

$$[\boldsymbol{A}|\boldsymbol{b}] = \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 1 & 3 & 1 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} := [\boldsymbol{R}|\boldsymbol{d}]$$

First and third are pivot columns, second and fourth are free columns.

Note that $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Rx} = \boldsymbol{0}$

Let $x_2 = s$ and $x_4 = t$. Then $x_3 = -4x_4 = -4t$ and $x_1 = -3x_2 - 2x_4 = -3s - 2t$.

$\therefore$ The solutions to $\boldsymbol{Ax} = \boldsymbol{0}$ are

$$\boldsymbol{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \forall\, s, t \in \mathbb{R}.$$

Let the free variables $x_2 = 0 = x_4$. A particular solution to $\boldsymbol{Ax} = \boldsymbol{b} (\Leftrightarrow \boldsymbol{Rx} = \boldsymbol{d})$ is

$$\boldsymbol{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}.$$

Therefore, the complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \forall\, s, t \in \mathbb{R}.$$

(4) (15 pts) Consider the 2×3 real matrix,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}.$$

Find the bases and dimensions for the four subspaces: $C(\boldsymbol{A})$, $C(\boldsymbol{A}^\top)$, $N(\boldsymbol{A})$, $N(\boldsymbol{A}^\top)$.

Solution:

- $C(\boldsymbol{A})$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis, since $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \therefore\ \dim C(\boldsymbol{A}) = 1$
- $C(\boldsymbol{A}^\top)$: $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is a basis, since $\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad \therefore\ \dim C(\boldsymbol{A}^\top) = 1$
- $N(\boldsymbol{A})$: $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow x_1 + 2x_2 + 4x_3 = 0$

  One can check that $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ are a basis for $N(\boldsymbol{A})$.

  $\therefore\ \dim N(\boldsymbol{A}) = 2$

- $N(\boldsymbol{A}^\top)$: Note that $\boldsymbol{A}^\top = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$.

  $\boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{0} \Leftrightarrow y_1 + 2y_2 = 0$

  One can check that $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for $N(\boldsymbol{A}^\top)$.

  $\therefore \dim N(\boldsymbol{A}^\top) = 1$

(5) (10 pts) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and let r be the rank of $\boldsymbol{A}$. State the Fundamental Theorem of Linear Algebra, Part I and Part II.

Solution:

- **Part I:**

$$\dim C(\boldsymbol{A}^\top) = r \quad \text{and} \quad \dim N(\boldsymbol{A}) = n - r.$$
$$\dim C(\boldsymbol{A}) = r \quad \text{and} \quad \dim N(\boldsymbol{A}^\top) = m - r.$$

- **Part II:**

$$C(\boldsymbol{A}^\top)^\perp = N(\boldsymbol{A}) \quad \text{and} \quad C(\boldsymbol{A})^\perp = N(\boldsymbol{A}^\top).$$

(6) (10 pts) Let $\boldsymbol{V}$ be a subspace of $\mathbb{R}^n$ and $\boldsymbol{V}^\perp := \{\boldsymbol{x} \in \mathbb{R}^n | \ \boldsymbol{x}\cdot\boldsymbol{v} = 0, \ \forall \boldsymbol{v} \in \boldsymbol{V}\}$ be the orthogonal complement of $\boldsymbol{V}$. Show that $\boldsymbol{V}^\perp$ is a subspace of $\mathbb{R}^n$ and $\boldsymbol{V}\cap\boldsymbol{V}^\perp = \{\boldsymbol{0}\}$.

Proof:

- Claim: $\boldsymbol{V}^\perp$ is a subspace of $\mathbb{R}^n$

  (i) Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}^\perp$. Then $\boldsymbol{x}\cdot\boldsymbol{v} = 0$ and $\boldsymbol{y}\cdot\boldsymbol{v} = 0$, $\forall\, \boldsymbol{v} \in \boldsymbol{V}$.

  $\therefore\ (\boldsymbol{x}+\boldsymbol{y})\cdot\boldsymbol{v} = \boldsymbol{x}\cdot\boldsymbol{v} + \boldsymbol{y}\cdot\boldsymbol{v} = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}$

  $\therefore\ \boldsymbol{x}+\boldsymbol{y} \in \boldsymbol{V}^\perp$

  (ii) Let $\boldsymbol{x} \in \boldsymbol{V}^\perp$ and $\alpha \in \mathbb{R}$. Then $\boldsymbol{x}\cdot\boldsymbol{v} = 0$, $\forall\, \boldsymbol{v} \in \boldsymbol{V}$.

  $\therefore\ (\alpha\boldsymbol{x})\cdot\boldsymbol{v} = \alpha(\boldsymbol{x}\cdot\boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}$

  $\therefore\ \alpha\boldsymbol{x} \in \boldsymbol{V}^\perp$

- Claim: $\boldsymbol{V}\cap\boldsymbol{V}^\perp = \{\boldsymbol{0}\}$

  $\because\ \boldsymbol{V}$ and $\boldsymbol{V}^\perp$ are subspaces of $\mathbb{R}^n$

  $\therefore\ \boldsymbol{0} \in \boldsymbol{V}$ and $\boldsymbol{0} \in \boldsymbol{V}^\perp$

  $\therefore\ \boldsymbol{0} \in \boldsymbol{V}\cap\boldsymbol{V}^\perp$

  Suppose that $\exists\, \boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{x} \in \boldsymbol{V}\cap\boldsymbol{V}^\perp$.

  Then $\boldsymbol{x} \in \boldsymbol{V}$ and $\boldsymbol{x} \in \boldsymbol{V}^\perp$

  $\therefore\ \boldsymbol{x}\cdot\boldsymbol{x} = 0 \Rightarrow \ \|\boldsymbol{x}\|^2 = 0 \Rightarrow \ \|\boldsymbol{x}\| = 0 \Rightarrow \ \boldsymbol{x} = \boldsymbol{0}$, a contradiction!

  $\therefore\ \boldsymbol{V}\cap\boldsymbol{V}^\perp = \{\boldsymbol{0}\}$

(7) (10 pts) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. Show that the nullspace $N(\boldsymbol{A})$ is the orthogonal complement of the row space $C(\boldsymbol{A}^\top)$ in $\mathbb{R}^n$, i.e., $C(\boldsymbol{A}^\top)^\perp = N(\boldsymbol{A})$.

Proof:

($\subseteq$): Let $\boldsymbol{x} \in C(\boldsymbol{A}^\top)^\perp$.

Then $\boldsymbol{x}$ is orthogonal to every vectors in in $C(\boldsymbol{A}^\top)$.

$\therefore\ \boldsymbol{x}$ is orthogonal to every rows of $\boldsymbol{A}$

$\therefore\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \ \Rightarrow\ \boldsymbol{x} \in N(\boldsymbol{A})$

($\supseteq$): Let $\boldsymbol{x} \in N(\boldsymbol{A})$. then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.

$\therefore\ \boldsymbol{x}$ is orthogonal to every rows of $\boldsymbol{A}$

$\therefore\ \boldsymbol{x}$ is orthogonal to every columns of $\boldsymbol{A}^\top$

$\therefore\ \boldsymbol{x} \in C(\boldsymbol{A}^\top)^\perp$

(8) Assume that $\boldsymbol{S}$ and $\boldsymbol{T}$ are two subspaces of the finite-dimensional vector space $(\boldsymbol{V}, \mathbb{R})$.

(a) (10 pts) Show that $\boldsymbol{S} \cap \boldsymbol{T}$ and $\boldsymbol{S} + \boldsymbol{T} := \{s + t \mid s \in \boldsymbol{S}, t \in \boldsymbol{T}\}$ are both subspaces of $\boldsymbol{V}$.

(b) (10 pts) Assume that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r\}$ is a basis for $\boldsymbol{S} \cap \boldsymbol{T}$, $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s\}$ is a basis for $\boldsymbol{S}$, and $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t\}$ is a basis for $\boldsymbol{T}$. Show that $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t\}$ is a basis for $\boldsymbol{S} + \boldsymbol{T}$. Hence, $\dim(\boldsymbol{S} + \boldsymbol{T}) = \dim\boldsymbol{S} + \dim\boldsymbol{T} - \dim(\boldsymbol{S} \cap \boldsymbol{T})$.

Proof:

(a)

- Claim: $\boldsymbol{S} \cap \boldsymbol{T}$ is a subspace of $(\boldsymbol{V}, \mathbb{R})$
  (i) Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S} \cap \boldsymbol{T}$. Then $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{S}$ and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{T}$.
  $\because$ $\boldsymbol{S}$ and $\boldsymbol{T}$ are two subspaces of $\boldsymbol{V}$
  $\therefore$ $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{S}$ and $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{T}$
  $\therefore$ $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{S} \cap \boldsymbol{T}$
  (ii) Let $\boldsymbol{x} \in \boldsymbol{S} \cap \boldsymbol{T}$ and $\alpha \in \mathbb{R}$. Then $\boldsymbol{x} \in \boldsymbol{S}$ and $\boldsymbol{x} \in \boldsymbol{T}$.
  $\because$ $\boldsymbol{S}$ and $\boldsymbol{T}$ are two subspaces of $\boldsymbol{V}$
  $\therefore$ $\alpha\boldsymbol{x} \in \boldsymbol{S}$ and $\alpha\boldsymbol{x} \in \boldsymbol{T}$
  $\therefore$ $\alpha\boldsymbol{x} \in \boldsymbol{S} \cap \boldsymbol{T}$
- Claim: $\boldsymbol{S} + \boldsymbol{T}$ is a subspace of $(\boldsymbol{V}, \mathbb{R})$
  (i) Let $\boldsymbol{x}_1 + \boldsymbol{y}_1, \boldsymbol{x}_2 + \boldsymbol{y}_2 \in \boldsymbol{S} + \boldsymbol{T}$, where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \boldsymbol{S}$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \boldsymbol{T}$.
  $\because$ $\boldsymbol{S}$ and $\boldsymbol{T}$ are two subspaces of $\boldsymbol{V}$
  $\therefore$ $\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \boldsymbol{S}$ and $\boldsymbol{y}_1 + \boldsymbol{y}_2 \in \boldsymbol{T}$
  $\therefore$ $(\boldsymbol{x}_1 + \boldsymbol{y}_1) + (\boldsymbol{x}_2 + \boldsymbol{y}_2) = (\boldsymbol{x}_1 + \boldsymbol{x}_2) + (\boldsymbol{y}_1 + \boldsymbol{y}_2) \in \boldsymbol{S} + \boldsymbol{T}$
  (ii) Let $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{S} + \boldsymbol{T}$ and $\alpha \in \mathbb{R}$. Then $\boldsymbol{x} \in \boldsymbol{S}$ and $\boldsymbol{y} \in \boldsymbol{T}$.
  $\because$ $\boldsymbol{S}$ and $\boldsymbol{T}$ are two subspaces of $\boldsymbol{V}$
  $\therefore$ $\alpha\boldsymbol{x} \in \boldsymbol{S}$ and $\alpha\boldsymbol{y} \in \boldsymbol{T}$
  $\therefore$ $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y} \in \boldsymbol{S} + \boldsymbol{T}$

(b)

- $\text{span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t\} = \boldsymbol{S} + \boldsymbol{T}$
  ($\subseteq$): trivial!
  ($\supseteq$): Let $\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{S} + \boldsymbol{T}$. Then $\boldsymbol{x} \in \boldsymbol{S}$ and $\boldsymbol{y} \in \boldsymbol{T}$.
  $\because$ $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s\}$ is a basis for $\boldsymbol{S}$
  $\therefore$ $\exists\, \alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s \in \mathbb{R}$ s.t. $\boldsymbol{x} = \alpha_1\boldsymbol{u}_1 + \cdots + \alpha_r\boldsymbol{u}_r + \beta_1\boldsymbol{v}_1 + \cdots + \beta_s\boldsymbol{v}_s$
  $\because$ $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t\}$ is a basis for $\boldsymbol{T}$
  $\therefore$ $\exists\, \gamma_1, \cdots, \gamma_r, \delta_1, \cdots, \delta_t \in \mathbb{R}$ s.t. $\boldsymbol{y} = \gamma_1\boldsymbol{u}_1 + \cdots + \gamma_r\boldsymbol{u}_r + \delta_1\boldsymbol{w}_1 + \cdots + \delta_t\boldsymbol{w}_t$
  $\therefore$ $\boldsymbol{x} + \boldsymbol{y} = \alpha_1\boldsymbol{u}_1 + \cdots + \alpha_r\boldsymbol{u}_r + \beta_1\boldsymbol{v}_1 + \cdots + \beta_s\boldsymbol{v}_s + \gamma_1\boldsymbol{u}_1 + \cdots + \gamma_r\boldsymbol{u}_r + \delta_1\boldsymbol{w}_1 + \cdots + \delta_t\boldsymbol{w}_t$
  $= (\alpha_1 + \gamma_1)\boldsymbol{u}_1 + \cdots + (\alpha_r + \gamma_r)\boldsymbol{u}_r + \beta_1\boldsymbol{v}_1 + \cdots + \beta_s\boldsymbol{v}_s + \delta_1\boldsymbol{w}_1 + \cdots + \delta_t\boldsymbol{w}_t$
  $\in \text{span}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t\}$
- $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t$ are linearly independent
  Let $\alpha_1\boldsymbol{u}_1 + \cdots + \alpha_r\boldsymbol{u}_r + \beta_1\boldsymbol{v}_1 + \cdots + \beta_s\boldsymbol{v}_s + \delta_1\boldsymbol{w}_1 + \cdots + \delta_t\boldsymbol{w}_t = \boldsymbol{0}$.
  Then $\underbrace{\alpha_1\boldsymbol{u}_1 + \cdots + \alpha_r\boldsymbol{u}_r + \beta_1\boldsymbol{v}_1 + \cdots + \beta_s\boldsymbol{v}_s}_{:=\boldsymbol{x}} = \underbrace{-\delta_1\boldsymbol{w}_1 - \cdots - \delta_t\boldsymbol{w}_t}_{:=-\boldsymbol{y}}$.
  $\therefore$ $\boldsymbol{y} = \delta_1\boldsymbol{w}_1 + \cdots + \delta_t\boldsymbol{w}_t = -\boldsymbol{x} \in \boldsymbol{S}$ and $\boldsymbol{y} \in \boldsymbol{T}$
  $\therefore$ $\boldsymbol{y} \in \boldsymbol{S} \cap \boldsymbol{T}$
  $\therefore$ $\exists\, \gamma_1, \cdots, \gamma_r \in \mathbb{R}$ s.t. $\boldsymbol{y} = \gamma_1\boldsymbol{u}_1 + \cdots + \gamma_r\boldsymbol{u}_r$
  $\therefore$ $(\alpha_1 - \gamma_1)\boldsymbol{u}_1 + \cdots + (\alpha_r - \gamma_r)\boldsymbol{u}_r + \beta_1\boldsymbol{v}_1 + \cdots + \beta_s\boldsymbol{v}_s = \boldsymbol{0}$
  $\because$ $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s$ are linearly independent
  $\therefore$ $\beta_1 = 0, \cdots, \beta_s = 0$
  Similarly, we can prove that $\delta_1 = 0, \cdots, \delta_t = 0$.
  $\therefore$ $\alpha_1\boldsymbol{u}_1 + \cdots + \alpha_r\boldsymbol{u}_r = \boldsymbol{0}$
  $\because$ $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r\}$ is a basis for $\boldsymbol{S} \cap \boldsymbol{T}$
  $\therefore$ $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r$ are linearly independent
  $\therefore$ $\alpha_1 = 0, \cdots, \alpha_r = 0$
  $\therefore$ $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_r, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_s, \boldsymbol{w}_1, \cdots, \boldsymbol{w}_t$ are linearly independent