MA 2007B: Linear Algebra I
Complementary Note #1

(1) **Definition:** A norm on $\mathbb{R}^n$ is a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following three conditions:

(i) $f(\boldsymbol{v}) \geq 0, \forall\, \boldsymbol{v} \in \mathbb{R}^n$ and $f(\boldsymbol{v}) = 0 \Leftrightarrow \boldsymbol{v} = \mathbf{0}$.

(ii) $f(\alpha \boldsymbol{v}) = |\alpha| f(\boldsymbol{v}), \forall\, \boldsymbol{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

(iii) $f(\boldsymbol{v} + \boldsymbol{w}) \leq f(\boldsymbol{v}) + f(\boldsymbol{w}), \forall\, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. (called the triangle inequality)

(2) **Example:** Define a function f on $\mathbb{R}^n$ by

$$f(\boldsymbol{v}) := \|\boldsymbol{v}\|_\infty := \max\{|v_1|, |v_2|, \cdots, |v_n|\}, \quad \forall\, \boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n.$$

Then $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$.

Proof:

(i) (a). Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$. Then we have

$\because\ \|\boldsymbol{v}\|_\infty = \max\{|v_1|, |v_2|, \cdots, |v_n|\}$ and $|v_i| \geq 0, \forall\, i = 1, 2, \cdots, n$

$\therefore\ \|\boldsymbol{v}\|_\infty \geq 0,\ \forall\, \boldsymbol{v} \in \mathbb{R}^n$

(b). $\|\boldsymbol{v}\|_\infty = \mathbf{0} \Leftrightarrow \max\{|v_1|, |v_2|, \cdots, |v_n|\} = 0 \Leftrightarrow |v_i| = 0, \forall\, i = 1, 2, \cdots, n$
$\Leftrightarrow v_i = 0, \forall\, i = 1, 2, \cdots, n \Leftrightarrow \boldsymbol{v} = \mathbf{0}$

(ii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then we have

$$\begin{aligned}
\|\alpha \boldsymbol{v}\|_\infty &= \|(\alpha v_1, \alpha v_2 \cdots, \alpha v_n)\|_\infty = \max\{|\alpha v_1|, |\alpha v_2|, \cdots, |\alpha v_n|\} \\
&= \max\{|\alpha||v_1|, |\alpha||v_2|, \cdots, |\alpha||v_n|\} \\
&= |\alpha| \max\{|v_1|, |v_2|, \cdots, |v_n|\} \\
&= |\alpha| \|\boldsymbol{v}\|_\infty
\end{aligned}$$

(iii) Let $\boldsymbol{v} = (v_1, v_2, \cdots, v_n), \boldsymbol{w} = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
\|\boldsymbol{v} + \boldsymbol{w}\|_\infty &= \|(v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)\|_\infty \\
&= \max\{|v_1 + w_1|, |v_2 + w_2|, \cdots, |v_n + w_n|\} \\
&\leq \max\{|v_1| + |w_1|, |v_2| + |w_2|, \cdots, |v_n| + |w_n|\} \\
&\leq \max\{|v_1|, |v_2|, \cdots, |v_n|\} + \max\{|w_1|, |w_2|, \cdots, |w_n|\} \\
&= \|\boldsymbol{v}\|_\infty + \|\boldsymbol{w}\|_\infty
\end{aligned}$$

By (i), (ii), (iii), $f(\boldsymbol{v}) := \|\boldsymbol{v}\|_\infty$ is a norm on $\mathbb{R}^n$.

(3) **Definition:** Let $\|\cdot\|_*$ and $\|\cdot\|_{**}$ be two norms on $\mathbb{R}^n$. We say that $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are equivalent if there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|\boldsymbol{v}\|_* \leq \|\boldsymbol{v}\|_{**} \leq C_2 \|\boldsymbol{v}\|_*,\ \forall\, \boldsymbol{v} \in \mathbb{R}^n.$$

(4) **Example:** $\|\boldsymbol{v}\|_1 := |v_1| + |v_2|$ and $\|\boldsymbol{v}\|_2 := \sqrt{v_1^2 + v_2^2}$ for all $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on $\boldsymbol{R}^2$.

Proof: Let $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$.

$\because\ \|\boldsymbol{v}\|_1^2 := (|v_1| + |v_2|)^2 = v_1^2 + 2|v_1||v_2| + v_2^2 \geq v_1^2 + v_2^2 = \|\boldsymbol{v}\|_2^2$

$\therefore\ \|\boldsymbol{v}\|_1 \geq \|\boldsymbol{v}\|_2$

$\because\ 2|v_1||v_2| \leq v_1^2 + v_2^2$

$\therefore\ \|\boldsymbol{v}\|_1^2 := (|v_1| + |v_2|)^2 = v_1^2 + 2|v_1||v_2| + v_2^2 \leq 2(v_1^2 + v_2^2) = 2\|\boldsymbol{v}\|_2^2$

$\therefore\ \dfrac{1}{\sqrt{2}}\|\boldsymbol{v}\|_1 \leq \|\boldsymbol{v}\|_2$

$\therefore$ we have

$$\frac{1}{\sqrt{2}}\|\boldsymbol{v}\|_1 \leq \|\boldsymbol{v}\|_2 \leq \|\boldsymbol{v}\|_1, \quad \forall\, \boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2.$$

That is, $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on $\boldsymbol{R}^2$.

(5) **Note:** In (4), we also have

$$\|\boldsymbol{v}\|_2 \leq \|\boldsymbol{v}\|_1 \leq \sqrt{2}\|\boldsymbol{v}\|_2, \quad \forall\, \boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2.$$