MA 2007B: Linear Algebra I

Complementary Note #2

(1) **Definition:** A vector space $\boldsymbol{V}$ over a field $\mathbb{F}$ (e.g., $\mathbb{R}$ or $\mathbb{C}$) is a nonempty set together with two closed operations:

$$+ : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}, \text{called vector addition, i.e., } \boldsymbol{v} + \boldsymbol{w} \in \boldsymbol{V}, \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V};$$
$$\cdot : \mathbb{F} \times \boldsymbol{V} \to \boldsymbol{V}, \text{called scalar multiplication, i.e., } c\boldsymbol{v} := c \cdot \boldsymbol{v} \in \boldsymbol{V}, \forall\, \boldsymbol{v} \in \boldsymbol{V}, c \in \mathbb{F},$$

that satisfy the following 8 rules:

(i) $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}, \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$

(ii) $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}, \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$

(iii) $\exists\,!$ vector, called zero vector and denoted by $\mathbf{0}$, such that $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}, \forall\, \boldsymbol{v} \in \boldsymbol{V}$

(iv) For each $\boldsymbol{v} \in \boldsymbol{V}$, $\exists\,!$ vector, called inverse and denoted by $-\boldsymbol{v}$, such that $\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$

(v) $1\boldsymbol{v} = \boldsymbol{v}, \forall\, \boldsymbol{v} \in \boldsymbol{V}$, where 1 denotes the multiplicative identity in $\mathbb{F}$

(vi) $c_1(c_2\boldsymbol{v}) = (c_1c_2)\boldsymbol{v}, \forall\, \boldsymbol{v} \in \boldsymbol{V}$ and $c_1, c_2 \in \mathbb{F}$

(vii) $c(\boldsymbol{v} + \boldsymbol{w}) = c\boldsymbol{v} + c\boldsymbol{w}, \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ and $c \in \mathbb{F}$

(viii) $(c_1 + c_2)\boldsymbol{v} = c_1\boldsymbol{v} + c_2\boldsymbol{v}, \forall\, \boldsymbol{v} \in \boldsymbol{V}$ and $c_1, c_2 \in \mathbb{F}$

(2) **Note:** Let $(\boldsymbol{V}, \mathbb{F})$ be a vector space.

(i) $c\mathbf{0} = \mathbf{0}, \forall\, c \in \mathbb{F}$.

Proof:

$\because\ c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0} + c\mathbf{0}$ (by (1)(iii) & (1)(vii))

$\therefore \mathbf{0} = c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}) = c\mathbf{0} + (c\mathbf{0}) + (-c\mathbf{0})) = c\mathbf{0} + \mathbf{0} = c\mathbf{0}$

(by (1)(iv) & (1)(ii) & (1)(iii))

$\therefore\ c\mathbf{0} = \mathbf{0}$

(ii) $0\boldsymbol{v} = \mathbf{0}, \forall\, \boldsymbol{v} \in \boldsymbol{V}$.

Proof:

$\because\ 0\boldsymbol{v} = (0 + 0)\boldsymbol{v} = 0\boldsymbol{v} + 0\boldsymbol{v}$ (by (1)(viii))

$\therefore\ \mathbf{0} = (-0\boldsymbol{v}) + 0\boldsymbol{v} = (-0\boldsymbol{v}) + (0\boldsymbol{v} + 0\boldsymbol{v}) = (-0\boldsymbol{v} + 0\boldsymbol{v}) + 0\boldsymbol{v} = \mathbf{0} + 0\boldsymbol{v} = 0\boldsymbol{v}$

(by (1)(iv) & (1)(ii) & (1)(iii))

(iii) $-\boldsymbol{v} = (-1)\boldsymbol{v}$

Proof:

$\because\ \mathbf{0} = 0\boldsymbol{v} = (1 + (-1))\boldsymbol{v} = \boldsymbol{v} + (-1)\boldsymbol{v}$ (by (2)(ii) & (1)(viii) & (1)(iv))

$\therefore\ -\boldsymbol{v} = (-1)\boldsymbol{v}$

(3) **Example:** Let $\mathcal{M} := \mathbb{R}^{2\times 2} = \left\{ \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \middle| \, a_{ij} \in \mathbb{R}, i,j = 1,2 \right\}$ be the set of all 2×2 real matrices. Define two closed operations by

$$\begin{aligned} \boldsymbol{A}+\boldsymbol{B} &= \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] := \left[\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{array} \right] \in \mathcal{M}, \\ & \forall\, \boldsymbol{A} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right], \boldsymbol{B} = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] \in \mathcal{M}, \\ c\boldsymbol{A} &= c \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] := \left[\begin{array}{cc} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{array} \right] \in \mathcal{M}, \quad \forall\, \boldsymbol{A} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \in \mathcal{M},\ c \in \mathbb{R}. \end{aligned}$$

Proof:

(i) $\forall\, \boldsymbol{A} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right], \boldsymbol{B} = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] \in \mathcal{M}$, we have

$$\begin{aligned} \boldsymbol{A}+\boldsymbol{B} &= \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] = \left[\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{array} \right] \\ &= \left[\begin{array}{cc} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22} \end{array} \right] = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] + \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] = \boldsymbol{B}+\boldsymbol{A}. \end{aligned}$$

(ii) $\forall\, \boldsymbol{A} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right], \boldsymbol{B} = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right], \boldsymbol{C} = \left[\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right] \in \mathcal{M}$, we have

$$\begin{aligned} \boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}) &= \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left(\left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] + \left[\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right] \right) \\ &= \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left[\begin{array}{cc} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{array} \right] \\ &= \left[\begin{array}{cc} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) \end{array} \right] \\ &= \left[\begin{array}{cc} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} \end{array} \right] \\ &= \left[\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{array} \right] + \left[\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right] \\ &= \left(\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right] \right) + \left[\begin{array}{cc} c_{11} & c_{12} \\ c_{21} & c_{22} \end{array} \right] = (\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}. \end{aligned}$$

(iii) Let $\boldsymbol{0} := \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]$. Then $\forall\, \boldsymbol{A} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \in \mathcal{M}$, we have

$$\boldsymbol{A}+\boldsymbol{0} = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] = \left[\begin{array}{cc} a_{11}+0 & a_{12}+0 \\ a_{21}+0 & a_{22}+0 \end{array} \right] = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] = \boldsymbol{A}.$$

If $\boldsymbol{0}^*$ is another vector in $\mathcal{M}$ such that $\boldsymbol{A}+\boldsymbol{0}^* = \boldsymbol{A}, \forall \boldsymbol{A} \in \mathcal{M}$. Then

$$\boldsymbol{0} = \boldsymbol{0}+\boldsymbol{0}^* = \boldsymbol{0}^*+\boldsymbol{0} = \boldsymbol{0}^*.$$

Therefore, the zero vector is unique!

(iv) For each $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}$, we define $-\boldsymbol{A} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} \in \mathcal{M}$. Then

$$\begin{aligned} \boldsymbol{A} + (-\boldsymbol{A}) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + (-a_{11}) & a_{12} + (-a_{12}) \\ a_{21} + (-a_{21}) & a_{22} + (-a_{22}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \boldsymbol{0}. \end{aligned}$$

If $\boldsymbol{B}$ is another vector such that $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{0}$, then by (iii) & (ii), we have

$$\begin{aligned} -\boldsymbol{A} &= -\boldsymbol{A} + \boldsymbol{0} = -\boldsymbol{A} + (\boldsymbol{A} + \boldsymbol{B}) = (-\boldsymbol{A} + \boldsymbol{A}) + \boldsymbol{B} = (\boldsymbol{A} + (-\boldsymbol{A})) + \boldsymbol{B} \\ &= \boldsymbol{0} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{0} = \boldsymbol{B}. \end{aligned}$$

Therefore, the inverse $-\boldsymbol{A}$ is unique!

(v) $\forall\, \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}$, we have

$$1\boldsymbol{A} = 1 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1a_{11} & 1a_{12} \\ 1a_{21} & 1a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \boldsymbol{A}.$$

(vi) $\forall\, c_1, c_2 \in \mathbb{R}$ and $\forall\, \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}$, we have

$$\begin{aligned} c_1(c_2\boldsymbol{A}) &= c_1 \begin{bmatrix} c_2a_{11} & c_2a_{12} \\ c_2a_{21} & c_2a_{22} \end{bmatrix} = \begin{bmatrix} c_1(c_2a_{11}) & c_1(c_2a_{12}) \\ c_1(c_2a_{21}) & c_1(c_2a_{22}) \end{bmatrix} \\ &= \begin{bmatrix} (c_1c_2)a_{11} & (c_1c_2)a_{12} \\ (c_1c_2)a_{21} & (c_1c_2)a_{22} \end{bmatrix} = (c_1c_2) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = (c_1c_2)\boldsymbol{A}. \end{aligned}$$

(vii) $\forall\, c \in \mathbb{R}$ and $\forall\, \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathcal{M}$, we have

$$\begin{aligned} c(\boldsymbol{A} + \boldsymbol{B}) &= c\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) = c \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \\ &= \begin{bmatrix} ca_{11} + cb_{11} & ca_{12} + cb_{12} \\ ca_{21} + cb_{21} & ca_{22} + cb_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} + \begin{bmatrix} cb_{11} & cb_{12} \\ cb_{21} & cb_{22} \end{bmatrix} \\ &= c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + c \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = c\boldsymbol{A} + c\boldsymbol{B}. \end{aligned}$$

(viii) $\forall\, c_1, c_2 \in \mathbb{R}$ and $\forall\, \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathcal{M}$, we have

$$\begin{aligned} (c_1 + c_2)\boldsymbol{A} &= \begin{bmatrix} (c_1 + c_2)a_{11} & (c_1 + c_2)a_{12} \\ (c_1 + c_2)a_{21} & (c_1 + c_2)a_{22} \end{bmatrix} = \begin{bmatrix} c_1a_{11} + c_2a_{11} & c_1a_{12} + c_2a_{12} \\ c_1a_{21} + c_2a_{21} & c_1a_{22} + c_2a_{22} \end{bmatrix} \\ &= \begin{bmatrix} c_1a_{11} & c_1a_{12} \\ c_1a_{21} & c_1a_{22} \end{bmatrix} + \begin{bmatrix} c_2a_{11} & c_2a_{12} \\ c_2a_{21} & c_2a_{22} \end{bmatrix} = c_1\boldsymbol{A} + c_2\boldsymbol{A}. \end{aligned}$$