MA 2007B: Linear Algebra I – Quiz #1

Name: **Student ID number:**

---

(1) (5 pts) Let $\|\boldsymbol{v}\|_1 := |v_1| + |v_2|$ for all $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$. Show that $f(\boldsymbol{v}) := \|\boldsymbol{v}\|_1$ is a norm on $\mathbb{R}^2$.

Proof:

(i) $\|\boldsymbol{v}\|_1 = |v_1| + |v_2| \geq 0, \forall\ \boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$.

$\|\boldsymbol{v}\|_1 = |v_1| + |v_2| = 0 \ \Leftrightarrow\ |v_1| = 0, |v_2| = 0 \ \Leftrightarrow\ v_1 = 0, v_2 = 0 \ \Leftrightarrow\ \boldsymbol{v} = \mathbf{0}$

(ii) $\|\alpha\boldsymbol{v}\|_1 = \|(\alpha v_1, \alpha v_2)\|_1 = |\alpha v_1| + |\alpha v_2| = |\alpha|\big(|v_1| + |v_2|\big) = |\alpha|\|\boldsymbol{v}\|_1,$

$\forall\ \boldsymbol{v} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

(iii) $\|\boldsymbol{v} + \boldsymbol{w}\|_1 = \|(v_1 + w_1, v_2 + w_2)\|_1 = |v_1 + w_1| + |v_2 + w_2|$

$\leq |v_1| + |w_1| + |v_2| + |w_2| = |v_1| + |v_2| + |w_1| + |w_2| = \|\boldsymbol{v}\|_1 + \|\boldsymbol{w}\|_1,$

$\forall\ \boldsymbol{v} = (v_1, v_2),\ \boldsymbol{w} = (w_1, w_2) \in \mathbb{R}^2.$

By (i), (ii), (iii), $f(\boldsymbol{v}) := \|\boldsymbol{v}\|_1$ is a norm on $\mathbb{R}^2$.

(2) (5 pts) Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two nonzero vectors in $\mathbb{R}^2$. Let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Show that $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2$.

Proof:

$$\begin{aligned}
\|\boldsymbol{v} - \boldsymbol{w}\|^2 &= (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) \\
&= \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} \\
&= \|\boldsymbol{v}\|^2 - 2\boldsymbol{v} \cdot \boldsymbol{w} + \|\boldsymbol{w}\|^2 \\
&= \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta + \|\boldsymbol{w}\|^2.
\end{aligned}$$