MA 2007B: Linear Algebra I – Quiz #4

Name: **Student ID number:**

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(1) (5 pts) Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be an $m \times n$ real matrix. Show that the column space $C(\boldsymbol{A})$ of matrix $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

Proof: Note that $C(\boldsymbol{A}) = \{\boldsymbol{A}\boldsymbol{x} | \ \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$.

Method 1 Let $\boldsymbol{Ax}, \boldsymbol{Ay} \in C(\boldsymbol{A})$ and $c \in \mathbb{R}$.

(i) $\because \ \boldsymbol{Ax} + \boldsymbol{Ay} = \boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y})$ and $\boldsymbol{x} + \boldsymbol{y} \in \mathbb{R}^n$
$\therefore \ \boldsymbol{Ax} + \boldsymbol{Ay} \in C(\boldsymbol{A})$

(ii) $\because \ c\boldsymbol{Ax} = \boldsymbol{A}(c\boldsymbol{x})$ and $c\boldsymbol{x} \in \mathbb{R}^n$
$\therefore \ c\boldsymbol{Ax} \in C(\boldsymbol{A})$

By (i) and (ii), $C(\boldsymbol{A})$ is a subspace of $\mathbb{R}^m$.

Method 2 Let $\boldsymbol{Ax}, \boldsymbol{Ay} \in C(\boldsymbol{A})$ and $c_1, c_2 \in \mathbb{R}$.

$\because \ c_1\boldsymbol{Ax} + c_2\boldsymbol{Ay} = \boldsymbol{A}(c_1\boldsymbol{x} + c_2\boldsymbol{y})$ and $c_1\boldsymbol{x} + c_2\boldsymbol{y} \in \mathbb{R}^n$

$\therefore \ c_1\boldsymbol{Ax} + c_2\boldsymbol{Ay} \in \mathbb{R}^m$

$\therefore \ C(\boldsymbol{A})$ is a subspace of $\mathbb{R}^m$

(2) (5 pts) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}$. Find the nullspace space $N(\boldsymbol{A})$ of matrix $\boldsymbol{A}$.

Solution: Note that $N(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^4 | \ \boldsymbol{Ax} = \boldsymbol{0}\}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \xrightarrow{\ell_{21}=\ell_{31}=\frac{1}{1}} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{\ell_{32}=\frac{2}{1}} \begin{bmatrix} \textcircled{1} & 1 & 2 & 4 \\ 0 & \textcircled{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and second are pivot columns; the third and fourth are free columns. Therefore, x_3 and x_4 are free variables.

Let $x_3 = s$ and $x_4 = t$. Then $\boldsymbol{Ax} = \boldsymbol{0} \implies x_2 = -x_4 = -t$

$\implies x_1 = -x_2 - 2x_3 - 4x_4 = t - 2s - 4t = -2s - 3t.$

$$\therefore \ \boldsymbol{x} = \begin{bmatrix} -2s-3t \\ -t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ -t \\ 0 \\ t \end{bmatrix} = s \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{:=\boldsymbol{s}_1} + t \underbrace{\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{:=\boldsymbol{s}_2} = s\boldsymbol{s}_1 + t\boldsymbol{s}_2$$

$\therefore$ The nullspace $N(\boldsymbol{A})$ of matrix $\boldsymbol{A}$ is

$$N(\boldsymbol{A}) = \{s\boldsymbol{s}_1 + t\boldsymbol{s}_2 | \ s, t \in \mathbb{R}\}$$