MA 2007B: Linear Algebra I – Quiz #5

Name: **Student ID number:**

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(1) (5 pts) Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix}$. Find the complete solution to the linear system.

Solution:

$$[\boldsymbol{A}|\boldsymbol{b}] = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & -0 \end{bmatrix} := [\boldsymbol{R}|\boldsymbol{d}]$$

First and third are pivot columns, second and fourth are free columns.

Note that $\boldsymbol{Ax} = \boldsymbol{0} \Leftrightarrow \boldsymbol{Rx} = \boldsymbol{0}$

Let $x_2 = s$ and $x_4 = t$. Then $x_3 = -4x_4 = -4t$ and $x_1 = -3x_2 - 2x_4 = -3s - 2t$.

$\therefore$ The solutions to $\boldsymbol{Ax} = \boldsymbol{0}$ are

$$\boldsymbol{x}_n = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2t \\ 0 \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \forall\, s, t \in \mathbb{R}.$$

Let the free variables $x_2 = 0 = x_4$. A particular solution to $\boldsymbol{Ax} = \boldsymbol{b} (\Leftrightarrow \boldsymbol{Rx} = \boldsymbol{d})$ is

$$\boldsymbol{x}_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \end{bmatrix}.$$ Therefore, the complete solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \forall\, s, t \in \mathbb{R}.$$

(2) (5 pts) Show that $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are a basis for the vector space $\mathbb{R}^2$.

Proof:

(a) **Claim:** $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent:

If $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow c_1 = 0, c_2 = 0.$

(b) **Claim:** $\text{span}\{v_1, v_2\} = \mathbb{R}^2$:

For any $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, we have $\begin{bmatrix} x \\ y \end{bmatrix} = (x-y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\implies \mathbb{R}^2 \subseteq \text{span}\{v_1, v_2\}$.

Obviously, $\text{span}\{v_1, v_2\} \subseteq \mathbb{R}^2$. Therefore, $\text{span}\{v_1, v_2\} = \mathbb{R}^2$.