MA 2007B: Linear Algebra I – Quiz #7

Name: **Student ID number:**

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(1) Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. Show that $\boldsymbol{A}^\top\boldsymbol{A}$ has the same nullspace as $\boldsymbol{A}$, i.e., $N(\boldsymbol{A}^\top\boldsymbol{A}) = N(\boldsymbol{A})$.

Proof: Note that

$N(\boldsymbol{A}) := \{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$,

$N(\boldsymbol{A}^\top\boldsymbol{A}) := \{\boldsymbol{x} \in \mathbb{R}^n : \ \boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$.

(i) If $\boldsymbol{x} \in N(\boldsymbol{A})$, then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^\top\boldsymbol{0} = \boldsymbol{0} \implies \boldsymbol{x} \in N(\boldsymbol{A}^\top\boldsymbol{A})$

(ii) If $\boldsymbol{x} \in N(\boldsymbol{A}^\top\boldsymbol{A})$, then $\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x}^\top\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^\top\boldsymbol{0} = 0 \implies (\boldsymbol{A}\boldsymbol{x})^\top\boldsymbol{A}\boldsymbol{x} = 0$ $\implies \|\boldsymbol{A}\boldsymbol{x}\|^2 = 0 \implies \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0} \implies \boldsymbol{x} \in N(\boldsymbol{A})$

By (i) and (ii), $N(\boldsymbol{A}^\top\boldsymbol{A}) = N(\boldsymbol{A})$.

(2) Find the closest line $y = C + Dx$ to the points $(0,6), (1,0)$ and $(2,0)$ by solving the normal equation.

Solution: We have

$$\begin{cases} C + D\times 0 &= 6 \\ C + D\times 1 &= 0 \\ C + D\times 2 &= 0 \end{cases} \iff \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \iff \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable! Since

$$\boldsymbol{A}^\top\boldsymbol{A} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \text{ and } \boldsymbol{A}^\top\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \end{bmatrix},$$

the normal equation is

$$\boldsymbol{A}^\top\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^\top\boldsymbol{b} \iff \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \iff \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

The line is $y = 5 - 3x$.