MA 2007B: Linear Algebra I – Quiz #8

Name: **Student ID number:**

---

(1) Let $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ be two orthonormal vectors in $\mathbb{R}^m$. Show that $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are linearly independent.

Proof:

Let $\alpha \boldsymbol{q}_1 + \beta \boldsymbol{q}_2 = \mathbf{0}$.

Then $(\alpha \boldsymbol{q}_1 + \beta \boldsymbol{q}_2) \cdot \boldsymbol{q}_1 = \mathbf{0} \cdot \boldsymbol{q}_1 = 0$.

$\therefore\ \alpha \boldsymbol{q}_1 \cdot \boldsymbol{q}_1 + \beta \boldsymbol{q}_2 \cdot \boldsymbol{q}_1 = 0$

$\because\ \boldsymbol{q}_2 \cdot \boldsymbol{q}_1 = 0 \quad (\because$ they are orthonormal vectors$)$

$\therefore\ \alpha \boldsymbol{q}_1 \cdot \boldsymbol{q}_1 = 0$

$\because\ \boldsymbol{q}_1 \cdot \boldsymbol{q}_1 = \|\boldsymbol{q}_1\|^2 > 0 \quad (\because\ \boldsymbol{q}_1 \neq \mathbf{0})$

$\therefore\ \alpha = 0$

Similarly, $(\alpha \boldsymbol{q}_1 + \beta \boldsymbol{q}_2) \cdot \boldsymbol{q}_2 = \mathbf{0} \cdot \boldsymbol{q}_2 = 0$

We can prove $\beta = 0$.

$\therefore\ \boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are linearly independent

(2) Let $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$. Find the orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ by the Gram-Schmidt process.

Solution:

Let $A = \boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Then $B = \boldsymbol{b} - \dfrac{A^\top \boldsymbol{b}}{A^\top A} A = \boldsymbol{b} - \dfrac{0}{6} A = \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

$$C = \boldsymbol{c} - \frac{A^\top \boldsymbol{c}}{A^\top A} A - \frac{B^\top \boldsymbol{c}}{B^\top B} B = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \frac{9}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\therefore\ \boldsymbol{q}_1 = \frac{A}{\|A\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \quad \boldsymbol{q}_2 = \frac{B}{\|B\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad \boldsymbol{q}_3 = \frac{C}{\|C\|} = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$