MA3111: Mathematical Image Processing
Principal Component Pursuit

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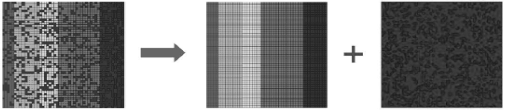
Sparse plus low rank matrix decomposition

Let $\boldsymbol{M} \in \mathbb{R}^{m \times n}$ be a given grayscale image. Suppose that $\boldsymbol{M}$ is the superposition of a low-rank component $\boldsymbol{L}$ and a sparse component $\boldsymbol{S}$,

$$\boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S}.$$

We are interested in finding the low-rank image $\boldsymbol{L}$, which has high repeatability along horizontal or vertical directions.



(schematic diagram)

The sparse plus low rank decomposition problem can be formulated as the constrained minimization problem:

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\operatorname{rank}(\boldsymbol{L}) + \lambda \|\boldsymbol{S}\|_0 \right) \quad \text{subject to} \quad \boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S},$$

where $\lambda > 0$ is a tuning parameter and $\|\boldsymbol{S}\|_0$ denotes the number of non-zero entries in $\boldsymbol{S}$. *The problem is not convex.*

The principal component pursuit problem

We approximate the sparse plus low rank decomposition problem by the following *principal component pursuit (PCP) problem:*

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \right) \quad \text{subject to} \quad \boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S},$$

where $\|\boldsymbol{L}\|_*$ is the nuclear (Ky Fan/樊"土畿") norm of $\boldsymbol{L}$ defined as

$$\|\boldsymbol{L}\|_* := \sum_{i=1}^{r} \sigma_i,$$

and $r \in \mathbb{N}^+$ is the rank of $\boldsymbol{L}$ and σ_i are the singular values of $\boldsymbol{L}$, and $\|\boldsymbol{S}\|_1$ denotes the ℓ^1-norm of $\boldsymbol{S}$ *(seen as a long vector in $\mathbb{R}^{mn}$),*

$$\|\boldsymbol{S}\|_1 := \sum_{i,j} |\boldsymbol{S}_{ij}|.$$

$\star$ *How about the existence of solution for the PCP problem?*
(cf. Candès-Li-Ma-Wright, J. ACM, 2011)

The penalty formulation and alternating direction method

Let $\mu > 0$ be the penalty parameter. Then we consider the relaxation using a penalty term to replace the constraint,

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L} - \boldsymbol{S}\|_F^2 \right),$$

where $\|\cdot\|_F$ is the Frobenius norm. We set, for example, $\boldsymbol{S}^{(0)} = \mathbf{0}$. The ADM for the penalty formulation is given as follows: for $k \geq 0$, find

$$\begin{aligned} \boldsymbol{L}^{(k+1)} &= \arg\min_{\boldsymbol{L}} \left(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}^{(k)}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L} - \boldsymbol{S}^{(k)}\|_F^2 \right), \\ \boldsymbol{S}^{(k+1)} &= \arg\min_{\boldsymbol{S}} \left(\|\boldsymbol{L}^{(k+1)}\|_* + \lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L}^{(k+1)} - \boldsymbol{S}\|_F^2 \right). \end{aligned}$$

By further analysis given below (pp. 7-15), we can prove that

$$\begin{aligned} \boldsymbol{L}^{(k+1)} &= \mathrm{SVT}_{\frac{1}{\mu}} \left(\boldsymbol{M} - \boldsymbol{S}^{(k)} \right), \\ \boldsymbol{S}^{(k+1)} &= \mathrm{sign}(\boldsymbol{M} - \boldsymbol{L}^{(k+1)}) \odot \max\left\{ |\boldsymbol{M} - \boldsymbol{L}^{(k+1)}| - (\lambda/\mu),\ 0 \right\}, \end{aligned}$$

where $\odot$ is the Hadamard product (i.e., element-wise product).

SVD and SVT

- **Singular value decomposition (SVD)**

  Let $\boldsymbol{M} \in \mathbb{R}^{m\times n}$. The SVD of $\boldsymbol{M}$ is the factorization in the form

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top,$$

  where $\boldsymbol{U} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ are orthogonal matrices ($\boldsymbol{U}\boldsymbol{U}^\top = \boldsymbol{I}$ and $\boldsymbol{V}\boldsymbol{V}^\top = \boldsymbol{I}$) and $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times n}$ is diagonal with all non-negative entries called the singular values of $\boldsymbol{M}$.

- **Singular value thresholding (SVT)**

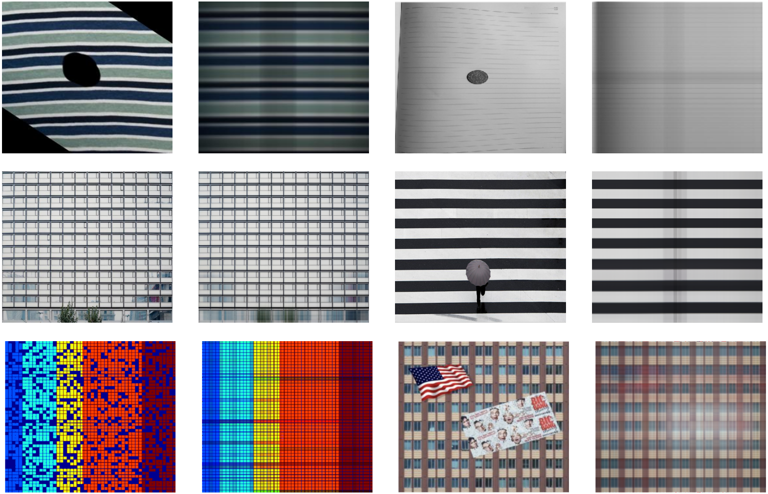
  Let $\boldsymbol{M} \in \mathbb{R}^{m\times n}$. Suppose that the SVD of $\boldsymbol{M}$ is given by $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$. Then the singular value thresholding (SVT) of $\boldsymbol{M}$ with threshold $\tau > 0$ is defined by

$$SVT_\tau(\boldsymbol{M}) = \boldsymbol{U}\boldsymbol{D}_\tau(\boldsymbol{\Sigma})\boldsymbol{V}^\top,$$

  where

$$\boldsymbol{D}_\tau(\boldsymbol{\Sigma})_{ii} = \max\{\boldsymbol{\Sigma}_{ii} - \tau,\ 0\}.$$

Background recovering using the penalty method



Von Neumann trace inequality

First, we state without proof the square matrix case.

Theorem: *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are complex $n \times n$ matrices with singular values*

$$\sigma_1(\boldsymbol{A}) \geq \sigma_2(\boldsymbol{A}) \geq \cdots \geq \sigma_n(\boldsymbol{A}) \geq 0,$$
$$\sigma_1(\boldsymbol{B}) \geq \sigma_2(\boldsymbol{B}) \geq \cdots \geq \sigma_n(\boldsymbol{B}) \geq 0.$$

Then we have

$$|\langle \boldsymbol{A}, \boldsymbol{B} \rangle_F| := |\mathrm{trace}(\boldsymbol{A}^* \boldsymbol{B})| \leq \sum_{i=1}^{n} \sigma_i(\boldsymbol{A}) \sigma_i(\boldsymbol{B}).$$

Moreover, the equality holds if $\boldsymbol{A}$ and $\boldsymbol{B}$ share the same singular vectors.

Notes:

- If $\boldsymbol{A} = \boldsymbol{U\Sigma V}^*$ then $\boldsymbol{A}^* = \boldsymbol{V\Sigma U}^*$, having the same singular values $\sigma_i(\boldsymbol{A}^*) = \sigma_i(\boldsymbol{A}), \forall\, 1 \leq i \leq n$. $\therefore$ $|\mathrm{trace}(\boldsymbol{AB})| \leq \sum_{i=1}^{n} \sigma_i(\boldsymbol{A}) \sigma_i(\boldsymbol{B})$.
- "Prove = if ...": If $\boldsymbol{A}$ and $\boldsymbol{B}$ share the same singular vectors, say $\boldsymbol{A} = \boldsymbol{U\Sigma}_A \boldsymbol{V}^*$ and $\boldsymbol{B} = \boldsymbol{U\Sigma}_B \boldsymbol{V}^*$, then we have
  $\boldsymbol{A}^*\boldsymbol{B} = \boldsymbol{V}(\boldsymbol{\Sigma}_A \boldsymbol{\Sigma}_B)\boldsymbol{V}^* = \boldsymbol{V}(\boldsymbol{\Sigma}_B \boldsymbol{\Sigma}_A)\boldsymbol{V}^* = \boldsymbol{B}^*\boldsymbol{A} = (\boldsymbol{A}^*\boldsymbol{B})^*$, Hermitian!
  $\therefore$ $\mathrm{trace}(\boldsymbol{A}^*\boldsymbol{B}) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{A}^*\boldsymbol{B}) = \sum_{i=1}^{n} \sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}) \geq 0$.

Von Neumann trace inequality for rectangular matrices

Corollary: *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be complex $m \times n$ matrices with singular values*

$$\sigma_1(\boldsymbol{A}) \geq \sigma_2(\boldsymbol{A}) \geq \cdots \geq \sigma_k(\boldsymbol{A}) \geq 0,$$
$$\sigma_1(\boldsymbol{B}) \geq \sigma_2(\boldsymbol{B}) \geq \cdots \geq \sigma_k(\boldsymbol{B}) \geq 0,$$

where $k := \min\{m, n\}$. Then we have

$$|\langle \boldsymbol{A}, \boldsymbol{B} \rangle_F| := |\mathrm{trace}(\boldsymbol{A}^* \boldsymbol{B})| \leq \sum_{i=1}^{k} \sigma_i(\boldsymbol{A}) \sigma_i(\boldsymbol{B}).$$

Moreover, the equality holds if $\boldsymbol{A}$ and $\boldsymbol{B}$ share the same singular vectors.

Proof: Assume that $m > n$. Then $k := \min\{m, n\} = n$. We define two $m \times m$ matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ by

$$\boldsymbol{X} = [\boldsymbol{A} \mid \mathbf{0}]_{m \times m} \quad \text{and} \quad \boldsymbol{Y} = [\boldsymbol{B} \mid \mathbf{0}]_{m \times m}.$$

Then we have

$$|\langle \boldsymbol{X}, \boldsymbol{Y} \rangle_F| = |\mathrm{trace}(\boldsymbol{X}^* \boldsymbol{Y})| = |\mathrm{trace}(\boldsymbol{A}^* \boldsymbol{B})| = |\langle \boldsymbol{A}, \boldsymbol{B} \rangle_F|.$$

Proof of Von Neumann's trace inequality (cont'd)

Claim: $\sigma_i(\boldsymbol{X}) = \sigma_i(\boldsymbol{A})$ and similarly, $\sigma_i(\boldsymbol{Y}) = \sigma_i(\boldsymbol{B})$, $\forall\, i = 1, 2, \cdots, n$.

Suppose that the SVD of $\boldsymbol{A}$ is given by $\boldsymbol{A}_{m\times n} = \boldsymbol{U}_{m\times m}\boldsymbol{\Sigma}_{m\times n}\boldsymbol{V}^*_{n\times n}$.
Define three $m \times m$ matrices,

$$\boldsymbol{U}_X = \boldsymbol{U}_{m\times m}, \quad \boldsymbol{\Sigma}_X = [\boldsymbol{\Sigma}_{m\times n} \mid \boldsymbol{0}]_{m\times m}, \quad \boldsymbol{V}_X^* = \begin{bmatrix} \boldsymbol{V}^*_{n\times n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}_{m\times m}.$$

Then

$$\begin{aligned} \boldsymbol{U}_X\boldsymbol{\Sigma}_X\boldsymbol{V}_X^* &= \boldsymbol{U}_{m\times m}[\boldsymbol{\Sigma}_{m\times n} \mid \boldsymbol{0}]\begin{bmatrix} \boldsymbol{V}^*_{n\times n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \\ &= [\boldsymbol{U}_{m\times m}\boldsymbol{\Sigma}_{m\times n} \mid \boldsymbol{0}]\begin{bmatrix} \boldsymbol{V}^*_{n\times n} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \\ &= [\boldsymbol{U}_{m\times m}\boldsymbol{\Sigma}_{m\times n}\boldsymbol{V}^*_{n\times n} \mid \boldsymbol{0}] = [\boldsymbol{A}_{m\times n} \mid \boldsymbol{0}] = \boldsymbol{X}, \end{aligned}$$

which implies that $\sigma_i(\boldsymbol{X}) = \sigma_i(\boldsymbol{A})$, $\forall\, i = 1, 2, \cdots, n$. Therefore,

$$|\langle \boldsymbol{A}, \boldsymbol{B}\rangle_F| = |\langle \boldsymbol{X}, \boldsymbol{Y}\rangle_F| \le \sum_{i=1}^{n} \sigma_i(\boldsymbol{X})\sigma_i(\boldsymbol{Y}) = \sum_{i=1}^{n} \sigma_i(\boldsymbol{A})\sigma_i(\boldsymbol{B}). \quad \square$$

$SVT_\tau(\boldsymbol{Y})$ Theorem

Theorem: *Given an $m \times n$ real matrix $\boldsymbol{Y}$ and $\tau > 0$, we have*

$$SVT_\tau(\boldsymbol{Y}) = \arg\min_{\boldsymbol{X} \in \mathbb{R}^{m\times n}} \left(\tau \|\boldsymbol{X}\|_* + \frac{1}{2}\|\boldsymbol{X} - \boldsymbol{Y}\|_F^2 \right).$$

Proof: Let $k := \min\{m, n\}$. Then for any $\boldsymbol{X} \in \mathbb{R}^{m\times n}$, we have

$$\begin{aligned}
\frac{1}{2}\|\boldsymbol{X} - \boldsymbol{Y}\|_F^2 &= \frac{1}{2}\mathrm{tr}((\boldsymbol{X} - \boldsymbol{Y})^\top(\boldsymbol{X} - \boldsymbol{Y})) \\
&= \frac{1}{2}\mathrm{tr}(\boldsymbol{X}^\top \boldsymbol{X}) - \mathrm{tr}(\boldsymbol{X}^\top \boldsymbol{Y}) + \frac{1}{2}\mathrm{tr}(\boldsymbol{Y}^\top \boldsymbol{Y}) \\
&= \frac{1}{2}\sum_{i=1}^{n} \lambda_i(\boldsymbol{X}^\top \boldsymbol{X}) + \frac{1}{2}\sum_{i=1}^{n} \lambda_i(\boldsymbol{Y}^\top \boldsymbol{Y}) - \mathrm{tr}(\boldsymbol{X}^\top \boldsymbol{Y}) \\
&\geq \frac{1}{2}\sum_{i=1}^{k} \sigma_i^2(\boldsymbol{X}) + \frac{1}{2}\sum_{i=1}^{k} \sigma_i^2(\boldsymbol{Y}) - \sum_{i=1}^{k} \sigma_i(\boldsymbol{X})\sigma_i(\boldsymbol{Y}) \\
&= \frac{1}{2}\sum_{i=1}^{k} \left(\sigma_i(\boldsymbol{X}) - \sigma_i(\boldsymbol{Y})\right)^2.
\end{aligned}$$

$SVT_\tau(\boldsymbol{Y})$ Theorem (cont'd)

Therefore, we obtain for any $\boldsymbol{X} \in \mathbb{R}^{m\times n}$,

$$F(\boldsymbol{X}) := \tau\|\boldsymbol{X}\|_* + \frac{1}{2}\|\boldsymbol{X}-\boldsymbol{Y}\|_F^2 \geq \tau\|\boldsymbol{X}\|_* + \frac{1}{2}\sum_{i=1}^{k}\big(\sigma_i(\boldsymbol{X})-\sigma_i(\boldsymbol{Y})\big)^2 =: G(\boldsymbol{X}).$$

It is already known that for a given $\tau > 0$ and a fixed $y \in \mathbb{R}$, the minimizer of the real-valued function,

$$f(x) = \tau|x| + \frac{1}{2}(y-x)^2, \quad x \in \mathbb{R},$$

is given by *the soft-thresholding operator* $\mathcal{S}_\tau$,

$$\arg\min_{x\in\mathbb{R}} f(x) = \mathcal{S}_\tau(y) := \operatorname{sign}(y)\max\{|y|-\tau,\ 0\}.$$

Also note that $\|\boldsymbol{X}\|_* = \sum_{i=1}^{k}\sigma_i(\boldsymbol{X})$. Therefore, we find the fact that

$$\begin{aligned}\widehat{\boldsymbol{X}} = \arg\min_{\boldsymbol{X}\in\mathbb{R}^{m\times n}} G(\boldsymbol{X}) \quad \Leftrightarrow \quad \sigma_i(\widehat{\boldsymbol{X}}) &= \mathcal{S}_\tau(\sigma_i(\boldsymbol{Y}))\\ &= \operatorname{sign}(\sigma_i(\boldsymbol{Y}))\max\{|\sigma_i(\boldsymbol{Y})|-\tau,\ 0\}\\ &= \max\{\sigma_i(\boldsymbol{Y})-\tau,\ 0\},\ \forall\, i = 1,2,\cdots,k.\end{aligned}$$

$SVT_\tau(\boldsymbol{Y})$ Theorem (cont'd)

Based on the above observation, we are going to construct such a matrix $\widehat{\boldsymbol{X}}$ which has the same singular vectors with $\boldsymbol{Y}$. Suppose that the SVD of $\boldsymbol{Y}$ is given by $\boldsymbol{Y} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$. Define the diagonal matrix $\widehat{\boldsymbol{\Sigma}}$ by

$$\widehat{\boldsymbol{\Sigma}} := \begin{bmatrix} \ddots & & \\ & \max\{\sigma_i(\boldsymbol{Y}) - \tau, 0\} & \\ & & \ddots \\ & & \end{bmatrix}_{m\times n}$$

and then define $\widehat{\boldsymbol{X}} := \boldsymbol{U}\widehat{\boldsymbol{\Sigma}}\boldsymbol{V}^\top = SVT_\tau(\boldsymbol{Y})$. Therefore, *the equality in Von Neumann's trace inequality holds,* and we have

$$\tau\|\widehat{\boldsymbol{X}}\|_* + \frac{1}{2}\|\widehat{\boldsymbol{X}} - \boldsymbol{Y}\|_F^2 = \tau\|\widehat{\boldsymbol{X}}\|_* + \frac{1}{2}\sum_{i=1}^{k}\left(\sigma_i(\widehat{\boldsymbol{X}}) - \sigma_i(\boldsymbol{Y})\right)^2 = \min_{\boldsymbol{X}\in\mathbb{R}^{m\times n}} G(\boldsymbol{X}).$$

That is, we attain a minimum of $F(\boldsymbol{X})$ at $\widehat{\boldsymbol{X}} = SVT_\tau(\boldsymbol{Y})$.

$F(\boldsymbol{X})$ is a strictly convex function in $\boldsymbol{X} \in \mathbb{R}^{m \times n}$

Note that $F(\boldsymbol{X})$ is a strictly convex function in $\boldsymbol{X} \in \mathbb{R}^{m \times n}$, since

- $\|\boldsymbol{X} - \boldsymbol{Y}\|_F^2$ is strictly convex in $\boldsymbol{X} \in \mathbb{R}^{m \times n}$.
- $\|\boldsymbol{X}\|_*$ is convex in $\boldsymbol{X} \in \mathbb{R}^{m \times n}$, since it is a norm.
- "convex function + strictly convex function" is strictly convex.

Suppose that $\widehat{\boldsymbol{X}}_1$ and $\widehat{\boldsymbol{X}}_2$ are two different minimizers of the strictly convex function $F(\boldsymbol{X})$. Then

$$F\left(\frac{1}{2}(\widehat{\boldsymbol{X}}_1 + \widehat{\boldsymbol{X}}_2)\right) < \frac{1}{2}F(\widehat{\boldsymbol{X}}_1) + \frac{1}{2}F(\widehat{\boldsymbol{X}}_2) = F(\widehat{\boldsymbol{X}}_1), \text{ a contradiction!}$$

Therefore, the minimizer of $F(\boldsymbol{X})$ is unique! This completes the proof of the theorem. $\square$

Another direct proof of the uniqueness of minimizer $\widehat{X}$

Claim: *The minimizer of $F(\boldsymbol{X})$ is unique, that is, $\widehat{\boldsymbol{X}} = SVT_\tau(\boldsymbol{Y})$.*

Proof: Suppose that $\widehat{\boldsymbol{X}}_1$ and $\widehat{\boldsymbol{X}}_2$ are two different minimizers of $F(\boldsymbol{X})$. By the triangle inequality, we have

$$\begin{aligned}
&\tau\|\frac{\widehat{\boldsymbol{X}}_1+\widehat{\boldsymbol{X}}_2}{2}\|_* + \frac{1}{2}\|\frac{\widehat{\boldsymbol{X}}_1+\widehat{\boldsymbol{X}}_2}{2} - \boldsymbol{Y}\|_F^2 \\
&\qquad\qquad \le \frac{\tau}{2}\|\widehat{\boldsymbol{X}}_1\|_* + \frac{\tau}{2}\|\widehat{\boldsymbol{X}}_2\|_* + \frac{1}{2}\|\frac{\widehat{\boldsymbol{X}}_1-\boldsymbol{Y}}{2} + \frac{\widehat{\boldsymbol{X}}_2-\boldsymbol{Y}}{2}\|_F^2. \qquad (\star)
\end{aligned}$$

Note that

$$\left(\frac{a}{2}+\frac{b}{2}\right)^2 = \frac{a^2}{2}+\frac{b^2}{2} - \left(\frac{a-b}{2}\right)^2, \quad \forall\, a,b \in \mathbb{R}.$$

Therefore, we obtain

$$\begin{aligned}
RHS(\star) \quad = \quad & \frac{\tau}{2}\|\widehat{\boldsymbol{X}}_1\|_* + \frac{\tau}{2}\|\widehat{\boldsymbol{X}}_2\|_* + \frac{1}{4}\|\widehat{\boldsymbol{X}}_1-\boldsymbol{Y}\|_F^2 + \frac{1}{4}\|\widehat{\boldsymbol{X}}_2-\boldsymbol{Y}\|_F^2 \\
& -\frac{1}{2}\|\frac{\widehat{\boldsymbol{X}}_1-\widehat{\boldsymbol{X}}_2}{2}\|_F^2 = \tau\|\widehat{\boldsymbol{X}}_1\|_* + \frac{1}{2}\|\widehat{\boldsymbol{X}}_1-\boldsymbol{Y}\|_F^2 - \underbrace{\frac{1}{2}\|\frac{\widehat{\boldsymbol{X}}_1-\widehat{\boldsymbol{X}}_2}{2}\|_F^2}_{>0},
\end{aligned}$$

a contradiction!

Solution of the ADM for penalty formulation

By the $SVT_\tau(\boldsymbol{Y})$ Theorem, we have

$$\boldsymbol{L}^{(k+1)} := \arg\min_{\boldsymbol{L}} \left(\|\boldsymbol{L}\|_* + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L} - \boldsymbol{S}^{(k)}\|_F^2 \right) = \mathrm{SVT}_{\frac{1}{\mu}} \left(\boldsymbol{M} - \boldsymbol{S}^{(k)} \right).$$

Using *the soft-thresholding operator* $\mathcal{S}_\tau$ again, we have

$$\begin{aligned} \boldsymbol{S}^{(k+1)} &:= \arg\min_{\boldsymbol{S}} \left(\lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L}^{(k+1)} - \boldsymbol{S}\|_F^2 \right) \\ &= \mathrm{sign}(\boldsymbol{M} - \boldsymbol{L}^{(k+1)}) \odot \max \left\{ |\boldsymbol{M} - \boldsymbol{L}^{(k+1)}| - (\lambda/\mu), \ 0 \right\}, \end{aligned}$$

where $\odot$ is the Hadamard element-wise product.

Another approach for solving the PCP problem

Recall the principal component pursuit problem:

$$\min_{L,S} \left(\|L\|_* + \lambda \|S\|_1 \right) \quad \text{subject to} \quad M = L + S.$$

The augmented Lagrangian function is defined as

$$\begin{aligned}
&\mathcal{L}(L, S, Y) \\
&:= \|L\|_* + \lambda \|S\|_1 + \langle \underbrace{Y}_{\text{multiplier}}, M - L - S \rangle + \underbrace{\frac{\mu}{2} \|M - L - S\|_F^2}_{\text{penalty}} \\
&= \|L\|_* + \lambda \|S\|_1 + \frac{\mu}{2} \|M - L - S + \mu^{-1} Y\|_F^2 - \frac{1}{2\mu} \|Y\|_F^2.
\end{aligned}$$

We then apply the alternating direction method to minimize the augmented Lagrangian function $\mathcal{L}(L, S, Y)$. The resulting method is called *the augmented Lagrange multiplier (ALM) method.*

The augmented Lagrange multiplier method

The ALM method is given by

$$
\begin{aligned}
\boldsymbol{L}^{(k+1)} &:= \arg\min_{\boldsymbol{L}} \Big(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}^{(k)}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L} - \boldsymbol{S}^{(k)} + \mu^{-1}\boldsymbol{Y}^{(k)}\|_F^2 \\
&\qquad - \frac{1}{2\mu}\|\boldsymbol{Y}^{(k)}\|_F^2 \Big), \\
\boldsymbol{S}^{(k+1)} &:= \arg\min_{\boldsymbol{S}} \Big(\|\boldsymbol{L}^{(k+1)}\|_* + \lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L}^{(k+1)} - \boldsymbol{S} + \mu^{-1}\boldsymbol{Y}^{(k)}\|_F^2 \\
&\qquad - \frac{1}{2\mu}\|\boldsymbol{Y}^{(k)}\|_F^2 \Big), \\
\boldsymbol{Y}^{(k+1)} &:= \boldsymbol{Y}^{(k)} + \mu\big(\boldsymbol{M} - \boldsymbol{L}^{(k+1)} - \boldsymbol{S}^{(k+1)}\big).
\end{aligned}
$$

The explicit form of the iterative solution $(\boldsymbol{L}^{(k+1)}, \boldsymbol{S}^{(k+1)}, \boldsymbol{Y}^{(k+1)})$ of ALM method is presented on the next page, which can be proved by using *the $SVT_\tau(\boldsymbol{Y})$ Theorem and the soft-thresholding operator S_τ.*

Iterative solutions of the ALM method

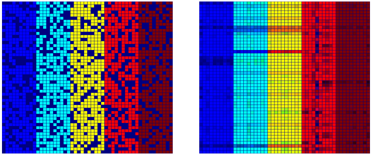
The iterative solution $(\boldsymbol{L}^{(k+1)}, \boldsymbol{S}^{(k+1)}, \boldsymbol{Y}^{(k+1)})$ of the ALM method is given by

$$\begin{aligned}
\boldsymbol{L}^{(k+1)} &:= \arg\min_{\boldsymbol{L}} \left(\|\boldsymbol{L}\|_* + \frac{\mu}{2} \|\boldsymbol{L} - (\boldsymbol{M} - \boldsymbol{S}^{(k)} + \mu^{-1}\boldsymbol{Y}^{(k)})\|_F^2 \right) \\
&= \arg\min_{\boldsymbol{L}} \left(\frac{1}{\mu} \|\boldsymbol{L}\|_* + \frac{1}{2} \|\boldsymbol{L} - (\boldsymbol{M} - \boldsymbol{S}^{(k)} + \mu^{-1}\boldsymbol{Y}^{(k)})\|_F^2 \right) \\
&= SVT_{\frac{1}{\mu}} (\boldsymbol{M} - \boldsymbol{S}^{(k)} + \mu^{-1}\boldsymbol{Y}^{(k)}), \\
\boldsymbol{S}^{(k+1)} &:= \arg\min_{\boldsymbol{S}} \left(\lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{S} - (\boldsymbol{M} - \boldsymbol{L}^{(k+1)} + \mu^{-1}\boldsymbol{Y}^{(k)})\|_F^2 \right) \\
&= \arg\min_{\boldsymbol{S}} \left(\frac{\lambda}{\mu} \|\boldsymbol{S}\|_1 + \frac{1}{2} \|\boldsymbol{S} - (\boldsymbol{M} - \boldsymbol{L}^{(k+1)} + \mu^{-1}\boldsymbol{Y}^{(k)})\|_F^2 \right) \\
&= \operatorname{sign}(\boldsymbol{M} - \boldsymbol{L}^{(k+1)} + \mu^{-1}\boldsymbol{Y}^{(k)}) \\
&\qquad \odot \max\left\{ |\boldsymbol{M} - \boldsymbol{L}^{(k+1)} + \mu^{-1}\boldsymbol{Y}^{(k)}| - (\lambda/\mu),\ 0 \right\}, \\
\boldsymbol{Y}^{(k+1)} &:= \boldsymbol{Y}^{(k)} + \mu\left(\boldsymbol{M} - \boldsymbol{L}^{(k+1)} - \boldsymbol{S}^{(k+1)}\right).
\end{aligned}$$

Background recovering using the ALM method

$(\lambda, \mu) = (0.0007, 0.5)$

$(\lambda, \mu) = (0.006, 5)$



$(\lambda, \mu) = (0.007525, 0.04)$

$(\lambda, \mu) = (0.0025, 1.5)$

References

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