MA3111: Mathematical Image Processing
Variational Image Segmentation

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Outline of "variational image segmentation"

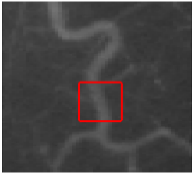
In this lecture, we will give a brief introduction to the topics:

- *The energy-based models for image segmentation: the Mumford-Shah model and the Chan-Vese model based on the level set formulation.*
- *An efficient iterative thresholding method for image segmentation.*
- *A local intensity clustering model for intensity inhomogeneous images.*

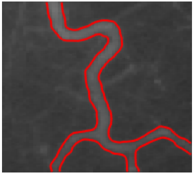
The material of this lecture is mainly based on

- P. Getreuer, Chan-Vese segmentation, *Image Processing On Line,* 2 (2012), pp. 214-224.
- D. Wang, H. Li, X. Wei, X.-P. Wang (王筱平), An efficient iterative thresholding method for image segmentation, *Journal of Computational Physics,* 350 (2017), pp. 657-667.
- C. Li (李純明), R. Huang, Z. Ding, J. C. Gatenby, D. N. Metaxas, and J. C. Gore, A level set method for image segmentation in the presence of intensity inhomogeneities with application to MRI, *IEEE Transactions on Image Processing,* 20 (2011), pp. 2007-2016.

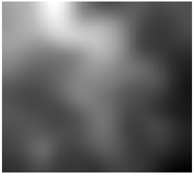
Image segmentation in medical imaging



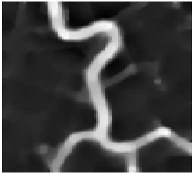
f & initialization C



segmented image



bias field b



corrected image I

Bias field model: $f = bI + n$, *where n is the noise*

In what follows, Ω denotes an open bounded subset in $\mathbb{R}^2$ and $f : \overline{\Omega} \to \mathbb{R}$ denotes the given grayscale image to be segmented.

Mumford-Shah model (CPAM 1989)

Mumford-Shah model: it finds a piecewise smooth function u and a curve set $\mathcal{C}$, which separates the image domain into disjoint regions, minimizing the energy functional:

$$\min_{u,\mathcal{C}} \left(\mu|\mathcal{C}| + \lambda \int_{\Omega} (f(\boldsymbol{x}) - u(\boldsymbol{x}))^2 \, d\boldsymbol{x} + \int_{\Omega \setminus \mathcal{C}} |\nabla u(\boldsymbol{x})|^2 \, d\boldsymbol{x} \right),$$

where $|\mathcal{C}|$ denotes the total length of the curves in $\mathcal{C}$.

- The first term plays the regularization role, which ensures the target objects can tightly be wrapped by $\mathcal{C}$.
- The second term is the data fidelity term, which forces u to be close to the input image f.
- The third term is the smoothing term, which forces the target function u to be piecewise smooth within each of the regions separated by the curves in $\mathcal{C}$.
- *$\mu > 0$, $\lambda > 0$ are tuning parameters to modulate these three terms.*

Simplified Mumford-Shah model

- *The non-convexity of energy functional in the Mumford-Shah model* makes the minimization problem difficult to analyze and the computational cost is much considerable.
- The piecewise smooth model suffers for its *sensitivity to the initialization of $\mathcal{C}$.*
- **Simplified Mumford-Shah model:** it finds *a piecewise constant function u* and a curve set $\mathcal{C}$ to minimize the energy functional:

$$\min_{u,\mathcal{C}} \left(\mu \left|\mathcal{C}\right| + \int_{\Omega} \left(f(\boldsymbol{x}) - u(\boldsymbol{x})\right)^2 d\boldsymbol{x} \right).$$

Note that u is constant on each connected component of $\Omega \setminus \mathcal{C}$. *The minimization problem is still non-convex.*

Chan (陳繁昌)-Vese two-phase model

In 1999, Chan and Vese proposed a two-phase segmentation model based on the level set formulation ("active contours without edges", LNCS 1999):

$$\min_{c_1,c_2,\mathcal{C}} \left(\mu|\mathcal{C}| + \nu|\Omega_{\text{in}}| + \lambda_1 \int_{\Omega_{\text{in}}} \left(f(\boldsymbol{x}) - c_1\right)^2 d\boldsymbol{x} + \lambda_2 \int_{\Omega_{\text{out}}} \left(f(\boldsymbol{x}) - c_2\right)^2 d\boldsymbol{x}\right),$$

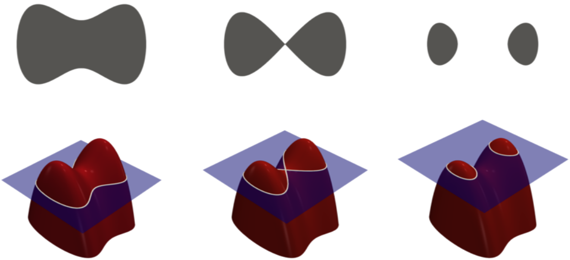
where

- Ω_{in} denotes the region enclosed by the curves in $\mathcal{C}$ with area $|\Omega_{\text{in}}|$, and $\Omega_{\text{out}} := \Omega \setminus \Omega_{\text{in}}$.
- $\mu > 0$, $\nu \geq 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ are tuning parameters (actually, one of them can be fixed as 1).
- Chan-Vese model finds a piecewise constant function u and a curve set $\mathcal{C}$ to minimize the energy functional, where u has only two constant values,

$$u(\boldsymbol{x}) = \begin{cases} c_1, & \boldsymbol{x} \text{ is inside } \mathcal{C}, \\ c_2, & \boldsymbol{x} \text{ is outside } \mathcal{C}. \end{cases}$$

Topological changes of $\mathcal{C}$

To solve the minimization problem of Chan-Vese model, we evolve $\mathcal{C}$ and find c_1, c_2 to minimize the energy functional. However, it is generally hard to handle topological changes of the curves in $\mathcal{C}$.



(quoted from wikipedia)

Level set function

Therefore, we represent $\mathcal{C}$ implicitly by the zero level contour of a level set function $\phi : \overline{\Omega} \to \mathbb{R}$, i.e.,

$$\mathcal{C} = \{\boldsymbol{x} \in \overline{\Omega} : \ \phi(\boldsymbol{x}) = 0\}.$$

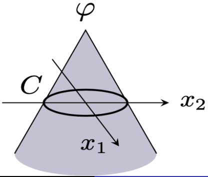
The zero level contour $\mathcal{C}$ partitions the image domain into two disjoint regions Ω_{in} and Ω_{out} such that

$$\phi(\boldsymbol{x}) \geq 0 \ \text{ for } \ \boldsymbol{x} \in \Omega_{\text{in}} \quad \text{and} \quad \phi(\boldsymbol{x}) < 0 \ \text{ for } \ \boldsymbol{x} \in \Omega_{\text{out}}.$$

For example, given $r > 0$, we define a level set function, which is a *signed distance function*,

$$\phi(\boldsymbol{x}) = \phi(x, y) = r - \sqrt{x^2 + y^2},$$

whose zero level contour is the circle of radius $r > 0$.



Chan-Vese model

- Let H denote the Heaviside function and δ the Dirac delta function. Then

$$H(s) = \begin{cases} 1 & s \geq 0, \\ 0 & s < 0, \end{cases} \quad \text{and} \quad \frac{d}{ds}H(s) = \delta(s).$$

- In terms of H, δ, and the level set function ϕ, the Chan-Vese model has the form

$$\begin{aligned} \min_{c1,\, c2,\, \phi} \Big(\mu \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|\, d\boldsymbol{x} + \nu \int_\Omega H(\phi(\boldsymbol{x}))\, d\boldsymbol{x} \\ +\lambda_1 \int_\Omega (f(\boldsymbol{x}) - c_1)^2 H(\phi(\boldsymbol{x}))\, d\boldsymbol{x} \\ +\lambda_2 \int_\Omega (f(\boldsymbol{x}) - c_2)^2 (1 - H(\phi(\boldsymbol{x})))\, d\boldsymbol{x} \Big). \end{aligned}$$

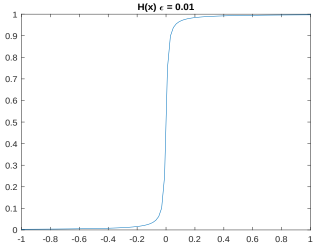
Original formulation:

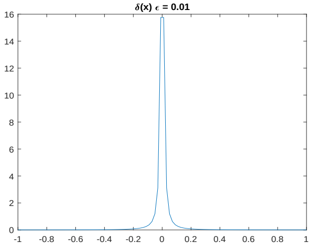
$$\min_{c_1, c_2, \mathcal{C}} \left(\mu|\mathcal{C}| + \nu|\Omega_{\text{in}}| + \lambda_1 \int_{\Omega_{\text{in}}} (f(\boldsymbol{x}) - c_1)^2 + \lambda_2 \int_{\Omega_{\text{out}}} (f(\boldsymbol{x}) - c_2)^2 \right).$$

The regularized Heaviside and delta functions

The Heaviside function H and the Dirac delta function δ can be approximately regularized as follows: for a sufficiently small $\epsilon > 0$,

$$
\begin{aligned}
H_\epsilon(t) &:= \frac{1}{2}\Big(1 + \frac{2}{\pi}\tan^{-1}\big(\frac{t}{\epsilon}\big)\Big),\\
\delta_\epsilon(t) &:= \frac{d}{dt}H_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2+t^2)},\\
\int_{-\infty}^{\infty}\delta_\epsilon(t)dt &= \int_{-\infty}^{\infty}\frac{\epsilon}{\pi(\epsilon^2+t^2)}dt = \cdots = 1.
\end{aligned}
$$





Total length of $\mathcal{C}$

The first term of the energy functional is the length of $\mathcal{C}$, which can be expressed as the total variation of $H(\phi)$,

$$\begin{aligned} |\mathcal{C}| &= \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} = \int_\Omega \left|\frac{dH}{d\phi}(\phi(\boldsymbol{x}))\right||\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \\ &= \int_\Omega |\nabla H(\phi(\boldsymbol{x}))|d\boldsymbol{x}. \end{aligned}$$

A heuristic argument to prove $|\mathcal{C}| = \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x}$:

Suppose that the level set function ϕ is a signed distance function, then we have $|\nabla\phi(\boldsymbol{x})| = 1$ for all $\boldsymbol{x} \in \overline{\Omega}$ (a.e.). The contour $\mathcal{C}$ can be parametrized in arc length s, $\boldsymbol{z}(s) = (x(s), y(s))$ for $0 \le s \le L := |\mathcal{C}|$. Let $N \gg 1$ be a large integer. We approximate the δ-function by

$$\delta_N(t) := \begin{cases} N, & |t| \le \frac{1}{2N}, \\ 0, & \text{otherwise.} \end{cases}$$

A heuristic argument (cont'd)

Let B_N be the narrow band defined by

$$B_N := \{\boldsymbol{x} \in \overline{\Omega} : \ |\phi(\boldsymbol{x})| \le 1/(2N)\}.$$

Then

$$\int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \approx N \int_{B_N} |\nabla\phi(\boldsymbol{x})|d\boldsymbol{x}.$$

The "centerline" of this band B_N is the curve $\mathcal{C} = \{\boldsymbol{x} \in \overline{\Omega} : \ \phi(\boldsymbol{x}) = 0\}$. Consider a point $\boldsymbol{p} = \boldsymbol{z}(s) \in \mathcal{C}$. Then the tangent vector and the normal vector are $\boldsymbol{z}'(s) = (x'(s), y'(s))$ and $\nabla\phi(\boldsymbol{z}(s))$, respectively. Starting at $\boldsymbol{p}$ in the direction $\nabla\phi(\boldsymbol{p})$, we reach the boundary of B_N when we have traversed the length $h > 0$ such that $|\nabla\phi(\boldsymbol{p})|h = \frac{1}{2N}$. It follows that near $\boldsymbol{p} = \boldsymbol{z}(s)$ the width $\rho(s)$ of this band is approximately given by

$$\rho(s) = 2h = \frac{1}{N|\nabla\phi(\boldsymbol{z}(s))|} = \frac{1}{N}.$$

Therefore we have

$$\int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \approx N \int_{B_N} |\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \approx N \int_0^L \rho(s)ds = L = |\mathcal{C}|.$$

An alternating iterative scheme

The minimization is solved by *an alternating iterative scheme,* i.e., alternatingly updating c_1, c_2 and ϕ.

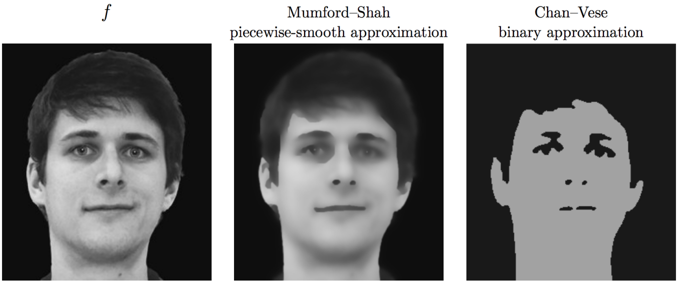
(S1) Fixed ϕ, the optimal values of c_1 and c_2 are the region averages,

$$c_1 = \frac{\int_\Omega f(\boldsymbol{x}) H(\phi(\boldsymbol{x}))\, d\boldsymbol{x}}{\int_\Omega H(\phi(\boldsymbol{x}))\, d\boldsymbol{x}}, \quad c_2 = \frac{\int_\Omega f(\boldsymbol{x}) \big(1 - H(\phi(\boldsymbol{x}))\big)\, d\boldsymbol{x}}{\int_\Omega \big(1 - H(\phi(\boldsymbol{x}))\big)\, d\boldsymbol{x}}.$$

(S2) Fixed c_1, c_2, we solve the initial-boundary value problem (IBVP) to reach a steady-state:

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= \delta_\epsilon(\phi)\Big(\mu \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} - \nu - \lambda_1 (f - c_1)^2 + \lambda_2 (f - c_2)^2\Big), \\
&\quad \text{for } \ t > 0, \boldsymbol{x} \in \Omega, \\
\phi(0, \boldsymbol{x}) &= \phi_0(\boldsymbol{x}), \boldsymbol{x} \in \Omega, \\
\frac{\partial \phi}{\partial \boldsymbol{n}} &= 0 \text{ on } \partial\Omega, t \geq 0.
\end{aligned}$$

Example: Mumford-Shah vs. Chan-Vese



P. Getreuer, Chan-Vese segmentation,
Image Processing On Line, 2 (2012), pp. 214-224.

Energy decreasing in time variable

The IBVP can also be derived by considering the decreasing of the Chan-Vese energy functional in time variable t.

(1) First, we introduce the time variable t and assume that the level set function ϕ evolves in time t, $\phi = \phi(t, x, y)$. Let $\Delta t > 0$ be an arbitrary small time step. *We suppose that the Chan-Vese energy functional is decreasing when the level set function ϕ evolves in time t.*

(2) For a given time $t \geq 0$, we define

$$\begin{aligned} v(x,y) &:= \frac{\partial \phi}{\partial t}(t,x,y)\Delta t \approx \phi(t+\Delta t, x, y) - \phi(t,x,y), \\ \psi(x,y) &:= \phi(t,x,y) + \alpha v(x,y) \approx \phi(t + \alpha \Delta t, x, y), \end{aligned}$$

where $0 < \alpha \ll 1$. Then $\psi_x = \phi_x + \alpha v_x$ and $\psi_y = \phi_y + \alpha v_y$.

(3) Let F be the integrand in the Chan-Vese energy functional. Then

$$\begin{aligned} E[\psi] &:= \int_\Omega F(x, y, \psi, \psi_x, \psi_y)\, d\mathbf{x}, \\ \left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} &= \int_\Omega \frac{\partial F}{\partial \phi} v + \frac{\partial F}{\partial \phi_x} v_x + \frac{\partial F}{\partial \phi_y} v_y \, d\mathbf{x} \leq 0. \end{aligned}$$

Energy decreasing in time variable (cont'd)

(4) Recall Green's formula,

$$\int_\Omega w \cdot \nabla p \, d\boldsymbol{x} = \int_{\partial\Omega} (w \cdot \boldsymbol{n}) p \, d\boldsymbol{\sigma} - \int_\Omega (\nabla \cdot w) p \, d\boldsymbol{x}.$$

Let $w = \left(\frac{\partial F}{\partial \phi_x}, \frac{\partial F}{\partial \phi_y}\right)$, $p = v$, and $\boldsymbol{n} = (n_1, n_2)$ be the unit normal vector to $\partial\Omega$. Then

$$\begin{aligned} \int_\Omega \frac{\partial F}{\partial \phi_x} v_x + \frac{\partial F}{\partial \phi_y} v_y \, d\boldsymbol{x} &= \int_{\partial\Omega} \left(\frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 \right) v \, d\boldsymbol{\sigma} \\ &\quad - \int_\Omega \left(\frac{\partial}{\partial x} \frac{\partial F}{\partial \phi_x} + \frac{\partial}{\partial y} \frac{\partial F}{\partial \phi_y} \right) v \, d\boldsymbol{x} \end{aligned}$$

Thus,

$$\begin{aligned} \left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} &= \int_\Omega \left\{ \frac{\partial F}{\partial \phi} v - \left(\frac{\partial}{\partial x} \frac{\partial F}{\partial \phi_x} \right) v - \left(\frac{\partial}{\partial y} \frac{\partial F}{\partial \phi_y} \right) v \right\} d\boldsymbol{x} \\ &\quad + \int_{\partial\Omega} \left(\frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 \right) v \, d\boldsymbol{\sigma}. \qquad (\star) \end{aligned}$$

Energy decreasing in time variable (cont'd)

Since

$$v(x,y) := \frac{\partial \phi}{\partial t}(t,x,y)\Delta t \approx \phi(t+\Delta t,x,y) - \phi(t,x,y),$$

it follows that $v(x,y) \approx 0$ for $(x,y) \in \partial\Omega$ and then

$$\left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} = \int_\Omega \left\{ \frac{\partial F}{\partial \phi} - \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial \phi_x}\right) - \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial \phi_y}\right) \right\} v\, d\mathbf{x} \le 0.$$

Therefore, we obtain *a sufficient condition* for $\left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} \le 0,$

$$\frac{\partial \phi}{\partial t}(t,x,y) = -\left\{ \frac{\partial F}{\partial \phi} - \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial \phi_x}\right) - \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial \phi_y}\right) \right\}.$$

Note that

$$\begin{aligned} F(x,y,\phi,\phi_x,\phi_y) &= \mu\delta_\epsilon(\phi)\,|\nabla\phi| + \nu H_\epsilon(\phi) + \lambda_1(f-c_1)^2 H_\epsilon(\phi) \\ &\quad + \lambda_2(f-c_2)^2(1-H_\epsilon(\phi)). \end{aligned}$$

Energy decreasing in time variable (cont'd)

By direct computations, we obtain

$$
\begin{aligned}
\frac{\partial F}{\partial \phi} &= \mu\delta'_\epsilon(\phi)\,|\nabla\phi| + \nu\delta_\epsilon(\phi) + \lambda_1(f-c_1)^2\delta_\epsilon(\phi) - \lambda_2(f-c_2)^2\delta_\epsilon(\phi), \\
\frac{\partial F}{\partial \phi_x} &= \mu\delta_\epsilon(\phi)\frac{\phi_x}{\sqrt{\phi_x^2+\phi_y^2}} = \mu\delta_\epsilon(\phi)\frac{\phi_x}{|\nabla\phi|}, \\
\frac{\partial F}{\partial \phi_y} &= \mu\delta_\epsilon(\phi)\frac{\phi_y}{|\nabla\phi|}.
\end{aligned}
$$

It leads to the equation

$$\frac{\partial\phi}{\partial t} = \delta_\epsilon(\phi)\left\{\mu\nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - \nu - \lambda_1(f-c_1)^2 + \lambda_2(f-c_2)^2\right\},$$

which has to be supplemented with an initial condition,

$$\phi(0,\boldsymbol{x}) = \phi_0(\boldsymbol{x}),\ \forall\ \boldsymbol{x}\in\Omega.$$

Neumann boundary condition

For a given time $t \geq 0$, if the energy functional E attains a local (or global) minimum at ϕ the we have

$$\int_{\partial\Omega} \left(\frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 \right) v \, d\boldsymbol{\sigma} = 0 \textit{ for any smooth function } v \textit{ on } \overline{\Omega}.$$

It follows that

$$0 = \frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 = \left(\frac{\partial F}{\partial \phi_x}, \frac{\partial F}{\partial \phi_y} \right) \cdot \boldsymbol{n} = \delta_\epsilon(\phi) \frac{\nabla \phi}{|\nabla \phi|} \cdot \boldsymbol{n}.$$

That is, we obtain the BC for $t \geq 0$,

$$\frac{\delta_\epsilon(\phi)}{|\nabla \phi|} \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega \quad \Longrightarrow \quad \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega$$

Numerical implementation

- Assume that the image domain $\overline{\Omega}$ is the unit square $[0,1] \times [0,1]$.
- Let $\Omega_D := \{(x_i, y_j) | \, i, j = 0, 1, \cdots, M\}$ be the set of grid points of a uniform partition of $\overline{\Omega}$ with size $h = 1/M$.
- Then $x_i = ih$ and $y_j = jh$, $i, j = 0, 1, \cdots, M$. Let $\phi_{i,j}(t)$ be the spatial difference approximation to $\phi(t, x_i, y_j)$.
- Let $t_n = n\Delta t$, $n \geq 0$, and $\Delta t > 0$ be the time step, and let $\phi^n_{i,j}$ be the full difference approximation to $\phi(t_n, x_i, y_j)$.

Discrete differential operators and BC

- *Define the discrete differential operators: for* $1 \le i, j \le M-1$,

$$
\begin{aligned}
\nabla_x^+ \phi_{i,j} &= \frac{\phi_{i+1,j} - \phi_{i,j}}{h}, \ \text{(forward difference)} \\
\nabla_x^- \phi_{i,j} &= \frac{\phi_{i,j} - \phi_{i-1,j}}{h}, \ \text{(backward difference)} \\
\nabla_y^+ \phi_{i,j} &= \frac{\phi_{i,j+1} - \phi_{i,j}}{h}, \ \text{(forward difference)} \\
\nabla_y^- \phi_{i,j} &= \frac{\phi_{i,j} - \phi_{i,j-1}}{h}, \ \text{(backward difference)} \\
\nabla_x^0 \phi_{i,j} &:= \left(\frac{\nabla_x^+ + \nabla_x^-}{2} \right) \phi_{i,j}, \quad \nabla_y^0 \phi_{i,j} := \left(\frac{\nabla_y^+ + \nabla_y^-}{2} \right) \phi_{i,j}. \\
& \quad \text{(central differences)}
\end{aligned}
$$

- *Discretize the homogeneous Neumann BC:* $\dfrac{\partial \phi}{\partial \mathbf{n}} = 0$ *on* $\partial \Omega$

$$
\phi_{0,j} = \phi_{1,j}, \quad \phi_{M,j} = \phi_{M-1,j}, \quad \phi_{i,0} = \phi_{i,1}, \quad \phi_{i,M} = \phi_{i,M-1}.
$$

Finite difference discretization: spatial variables

Performing the spatial discretization [Getreuer-2012], we have

$$
\begin{aligned}
\frac{\partial \phi_{i,j}}{\partial t} &= \delta_\epsilon(\phi_{i,j})\Bigg\{\mu\Bigg(\nabla_x^- \frac{\nabla_x^+\phi_{i,j}}{\sqrt{\eta^2+(\nabla_x^+\phi_{i,j})^2+(\nabla_y^0\phi_{i,j})^2}} \\
&\qquad +\nabla_y^- \frac{\nabla_y^+\phi_{i,j}}{\sqrt{\eta^2+(\nabla_x^0\phi_{i,j})^2+(\nabla_y^+\phi_{i,j})^2}}\Bigg) \\
&\qquad -\nu-\lambda_1(f_{i,j}-c_1)^2+\lambda_2(f_{i,j}-c_2)^2\Bigg\},
\end{aligned}
$$

where $i,j = 1,2,\cdots,M-1$.

The purpose of small positive parameter η in the denominators prevents division by zero.

Spatial discretization

Define

$$
\begin{aligned}
A_{i,j} &= \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^+ \phi_{i,j})^2 + (\nabla_y^0 \phi_{i,j})^2}}, \\
B_{i,j} &= \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^0 \phi_{i,j})^2 + (\nabla_y^+ \phi_{i,j})^2}}.
\end{aligned}
$$

Using the fact $\nabla_x^+ \phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{h}$, $\nabla_y^+ \phi_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{h}$ and taking the backward difference at $A_{i,j}(\phi_{i+1,j} - \phi_{i,j})$ and $B_{i,j}(\phi_{i,j+1} - \phi_{i,j})$, then the discretization can be written as

$$
\begin{aligned}
\frac{\partial \phi_{i,j}}{\partial t} = \delta_\epsilon(\phi_{i,j}) \Bigg\{ & \frac{1}{h^2}\Big(A_{i,j}(\phi_{i+1,j} - \phi_{i,j}) - A_{i-1,j}(\phi_{i,j} - \phi_{i-1,j}) \Big) \\
& + \frac{1}{h^2}\Big(B_{i,j}(\phi_{i,j+1} - \phi_{i,j}) - B_{i,j-1}(\phi_{i,j} - \phi_{i,j-1}) \Big) \\
& - \nu - \lambda_1 (f_{i,j} - c_1)^2 + \lambda_2 (f_{i,j} - c_2)^2 \Bigg\}.
\end{aligned}
$$

Temporal discretization

Define

$$\widetilde{A}_{i,j} = \frac{1}{h^2} A_{i,j}, \quad \widetilde{A}_{i-1,j} = \frac{1}{h^2} A_{i,j},$$
$$\widetilde{B}_{i,j} = \frac{1}{h^2} B_{i,j}, \quad \widetilde{B}_{i,j-1} = \frac{1}{h^2} B_{i,j-1}.$$

Time is discretized with a semi-implicit Gauss-Seidel method, values $\phi_{i,j}$, $\phi_{i-1,j}$, $\phi_{i,j-1}$ are evaluated at time t_{n+1} and all others at time t_n.

$$\begin{aligned} \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t} \quad = \quad & \delta_\epsilon(\phi_{i,j}^n) \Big\{ \widetilde{A}_{i,j}\phi_{i+1,j}^n + \widetilde{A}_{i-1,j}\phi_{i-1,j}^{n+1} + \widetilde{B}_{i,j}\phi_{i,j+1}^n + \widetilde{B}_{i,j-1}\phi_{i,j-1}^{n+1} \\ & -\left(\widetilde{A}_{i,j} + \widetilde{A}_{i-1,j} + \widetilde{B}_{i,j} + \widetilde{B}_{i,j-1}\right)\phi_{i,j}^{n+1} \\ & -\nu - \lambda_1(f_{i,j} - c_1)^2 + \lambda_2(f_{i,j} - c_2)^2 \Big\}. \end{aligned}$$

Gauss-Seidel scheme

This allows ϕ at time t_{n+1} to be solved by one Gauss-Seidel *sweep from left to right, bottom to top:*

$$\begin{aligned}
\phi_{i,j}^{n+1} &= \Big\{ \phi_{i,j}^n + \Delta t \delta_\epsilon(\phi_{i,j}^n) \Big(\widetilde{A}_{i,j}\phi_{i+1,j}^n + \widetilde{A}_{i-1,j}\phi_{i-1,j}^{n+1} + \widetilde{B}_{i,j}\phi_{i,j+1}^n \\
&\qquad + \widetilde{B}_{i,j-1}\phi_{i,j-1}^{n+1} - \nu - \lambda_1 (f_{i,j} - c_1)^2 + \lambda_2 (f_{i,j} - c_2)^2 \Big) \Big\} \\
&\qquad \times \Big\{ 1 + \Delta t \delta_\epsilon(\phi_{i,j}) \Big(\widetilde{A}_{i,j} + \widetilde{A}_{i-1,j} + \widetilde{B}_{i,j} + \widetilde{B}_{i,j-1} \Big) \Big\}^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{A}_{i,j} &= \frac{\mu}{h^2 \sqrt{\eta^2 + \Big((\phi_{i+1,j}^n - \phi_{i,j}^n)/h \Big)^2 + \Big((\phi_{i,j+1}^n - \phi_{i,j-1}^{n+1})/(2h) \Big)^2}}, \\
\widetilde{B}_{i,j} &= \frac{\mu}{h^2 \sqrt{\eta^2 + \Big((\phi_{i+1,j}^n - \phi_{i-1,j}^{n+1})/(2h) \Big)^2 + \Big((\phi_{i,j}^n - \phi_{i+1,j}^n)/h \Big)^2}}.
\end{aligned}$$

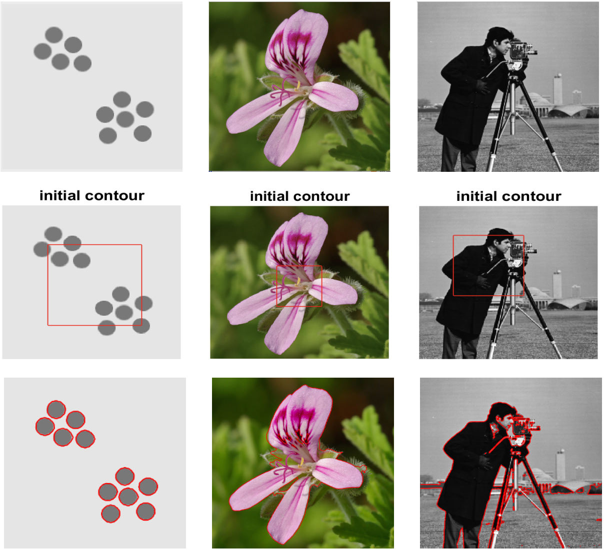
Gauss-Seidel scheme

We can rewrite $\widetilde{A}_{i,j}$ and $\widetilde{B}_{i,j}$ as follows:

$$\begin{aligned}
\widetilde{A}_{i,j} &= \frac{\mu}{h^2\sqrt{\eta^2 + \left(\left(\phi^n_{i+1,j} - \phi^n_{i,j}\right)/h\right)^2 + \left(\left(\phi^n_{i,j+1} - \phi^{n+1}_{i,j-1}\right)/(2h)\right)^2}}, \\
&= \frac{(\mu/h)}{\sqrt{(h\eta)^2 + (\phi^n_{i+1,j} - \phi^n_{i,j})^2 + \left(\left(\phi^n_{i,j+1} - \phi^{n+1}_{i,j-1}\right)/2\right)^2}}, \\
\widetilde{B}_{i,j} &= \frac{\mu}{h^2\sqrt{\eta^2 + \left(\left(\phi^n_{i+1,j} - \phi^{n+1}_{i-1,j}\right)/(2h)\right)^2 + \left(\left(\phi^n_{i,j} - \phi^n_{i+1,j}\right)/h\right)^2}} \\
&= \frac{(\mu/h)}{\sqrt{(h\eta)^2 + \left(\left(\phi^n_{i+1,j} - \phi^{n+1}_{i-1,j}\right)/2\right)^2 + (\phi^n_{i,j} - \phi^n_{i+1,j})^2}}.
\end{aligned}$$

In numerical implementation, we take $(h\eta) = 10^{-8}$.

Numerical experiments



The iterative convolution-thresholding scheme

- Most image segmentation models incorporate the level set formulation for solving the associated minimization problems. It generally results in initial-boundary value problems for PDEs.
- We are going to employ an *iterative convolution-thresholding (ICT) scheme* [WLWW-JCP2017] for multi-phase image segmentation based on the Chan-Vese model.
- In the ICT scheme, total length of $\mathcal{C}$ is approximated by a non-local multi-phase energy constructed based on *convolution of the heat kernel with the characteristic functions of regions.*
- The ICT scheme is divided into two steps. *It works by alternating a convolution step with the thresholding step.* The convolution can be implemented efficiently on a uniform mesh using the fast Fourier transform (FFT) with the optimal complexity of $O(N \log N)$ per iteration.

The approximate Chan-Vese functional

Let $f : \overline{\Omega} \to \mathbb{R}$ be the given grayscale image to be segmented.

- Suppose f approximately takes n distinct constants $c_1, \cdots, c_n$ in the disjoint regions $\Omega_1, \cdots, \Omega_n$ *(n-phase partition)* with boundaries $\mathcal{C}_1, \cdots, \mathcal{C}_n$, respectively, that separate Ω.

  Let $\mathcal{C} = \cup_{i=1}^{n} \mathcal{C}_i$. Then $\Omega \setminus \mathcal{C} = \cup_{i=1}^{n} \Omega_i$.

- Let χ_i be the characteristic function of the desirable region Ω_i,

$$\chi_i(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Omega_i, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \sum_{i=1}^{n} \chi_i = 1 \text{ in } \Omega \setminus \mathcal{C}.$$

- Let $\boldsymbol{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$. We define the set $\mathcal{S}$ of the characteristic vector functions by

$$\mathcal{S} = \left\{ \boldsymbol{\chi} \in (BV(\Omega))^n : \chi_i(\boldsymbol{x}) \in \{0, 1\}, \sum_{i=1}^{n} \chi_i(\boldsymbol{x}) = 1 \ \forall \boldsymbol{x} \in \Omega \setminus \mathcal{C} \right\},$$

  where $BV(\Omega)$ is the usual bounded variation space.

The approximate Chan-Vese functional (cont'd)

In [WLWW-JCP2017], the authors considered the following model:

$$\min_{\{\Omega_i\},\{c_i\}} \sum_{i=1}^{n} \left(\lambda |\mathcal{C}_i| + \int_{\Omega_i} (f(\boldsymbol{x}) - c_i)^2 \, d\boldsymbol{x} \right).$$

Let $\boldsymbol{c} := (c_1, c_2, \cdots, c_n)$. Then we look for $\boldsymbol{\chi}^*$ and $\boldsymbol{c}^*$ such that

$$(\boldsymbol{\chi}^*, \boldsymbol{c}^*) = \operatorname*{arg\,min}_{\boldsymbol{\chi} \in \mathcal{S}, \boldsymbol{c} \in \mathbb{R}^n} \sum_{i=1}^{n} \left(\lambda |\mathcal{C}_i| + \int_{\Omega} \chi_i(\boldsymbol{x}) g_i(\boldsymbol{x}) \, d\boldsymbol{x} \right),$$

where

$$g_i(\boldsymbol{x}) := (f(\boldsymbol{x}) - c_i)^2.$$

The length of $\mathcal{C}_i$

Let $0 < \tau \ll 1$. Define the heat kernel G_τ by

$$G_\tau(\boldsymbol{x}) := \frac{1}{4\pi\tau} \exp\Big(-\frac{\|\boldsymbol{x}\|_2^2}{4\tau}\Big).$$

Then the length of $\mathcal{C}_i \cap \mathcal{C}_j$ can be approximated by (see CPAM-2015)

$$|\mathcal{C}_i \cap \mathcal{C}_j| \approx \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x},$$

where $*$ represents the convolution operation, and therefore

$$|\mathcal{C}_i| \approx \sum_{j=1, j\neq i}^{n} \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x}.$$

S. Esedoḡlu and F. Otto, Threshold dynamics for networks with arbitrary surface tensions, *Communications on Pure and Applied Mathematics*, 68 (2015), pp. 808-864.

The approximate energy functional and ICT scheme

The total energy functional can be approximated by

$$\mathcal{E}_\tau(\chi, \boldsymbol{c}) = \sum_{i=1}^{n} \Big(\lambda \sum_{j=1, j\neq i}^{n} \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \chi_i(\boldsymbol{x}) g_i(\boldsymbol{x})\, d\boldsymbol{x} \Big),$$

and our goal is to solve the following minimization problem:

$$(\chi^*, \boldsymbol{c}^*) = \arg\min_{\chi \in \mathcal{S}, \boldsymbol{c} \in \mathbb{R}^n} \mathcal{E}_\tau(\chi, \boldsymbol{c}).$$

The minimization problem can be solved by the ICT scheme, i.e., alternatively updating χ and $\boldsymbol{c}$. Suppose that we have the k-th iterations for $k \geq 0$, $\chi^{(k)} = (\chi_1^{(k)}, \chi_2^{(k)}, \cdots, \chi_n^{(k)})$ and $\boldsymbol{c}^{(k)}$, then find $\chi^{(k+1)} \in \mathcal{S}$ and $\boldsymbol{c}^{(k+1)} \in \mathbb{R}^n$ sequentially such that

$$\begin{aligned} \chi^{(k+1)} &= \arg\min_{\chi \in \mathcal{S}} \mathcal{E}_\tau(\chi, \boldsymbol{c}^{(k)}), \\ \boldsymbol{c}^{(k+1)} &= \arg\min_{\boldsymbol{c} \in \mathbb{R}^n} \mathcal{E}_\tau(\chi^{(k+1)}, \boldsymbol{c}). \end{aligned}$$

The c-subproblem

Note that the energy functional is given by

$$\mathcal{E}_\tau(\chi, \boldsymbol{c}) = \sum_{i=1}^{n} \Big(\lambda \sum_{j=1, j\neq i}^{n} \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \chi_i(\boldsymbol{x}) g_i(\boldsymbol{x})\, d\boldsymbol{x} \Big).$$

Then

$$\min_{\boldsymbol{c}\in\mathbb{R}^n} \mathcal{E}_\tau(\chi^{(k+1)}, \boldsymbol{c}) = \min_{\boldsymbol{c}\in\mathbb{R}^n} \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})(f(\boldsymbol{x}) - c_i)^2\, d\boldsymbol{x}$$

Letting

$$\frac{\partial}{\partial c_i} \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})(f(\boldsymbol{x}) - c_i)^2\, d\boldsymbol{x} = 0,$$

we have

$$-2\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})(f(\boldsymbol{x}) - c_i)\, d\boldsymbol{x} = 0 \Longrightarrow c_i = \frac{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x}) f(\boldsymbol{x})\, d\boldsymbol{x}}{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})\, d\boldsymbol{x}}.$$

The χ-subproblem

Consider the χ-subproblem:

$$\chi^{(k+1)} = \arg\min_{\chi \in \mathcal{S}} \mathcal{E}_\tau(\chi, \boldsymbol{c}^{(k)}).$$

Note that the minimization problem is a non-convex problem since the characteristic function set $\mathcal{S}$ is not a convex set. In order to circumvent this drawback, we define the convex hull $\mathcal{K}$ of $\mathcal{S}$ by

$$\mathcal{K} = \left\{ \chi \in (BV(\Omega))^n : 0 \le \chi_i(\boldsymbol{x}) \le 1, \sum_{i=1}^{n} \chi_i(\boldsymbol{x}) = 1 \ \forall \boldsymbol{x} \in \Omega \setminus \mathcal{C} \right\}.$$

Then we consider the convex relaxed minimization problem instead:

$$\min_{\chi \in \mathcal{K}} \mathcal{E}_\tau(\chi, \boldsymbol{c}^{(k)}).$$

The χ-subproblem (cont'd)

In [WLWW-JCP2017], the authors proved that:

Assume that $\boldsymbol{\chi}^* \in \mathcal{K}$ *is a minimizer of* $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ *on* $\mathcal{K}$*, i.e.,*

$$\mathcal{E}_\tau(\boldsymbol{\chi}^*, \boldsymbol{c}^{(k)}) = \min_{\chi \in \mathcal{K}} \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}).$$

Then $\boldsymbol{\chi}^* \in \mathcal{S}$ *and hence it is also a minimizer of* $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ *on* $\mathcal{S}$*, i.e.,*

$$\mathcal{E}_\tau(\boldsymbol{\chi}^*, \boldsymbol{c}^{(k)}) = \min_{\chi \in \mathcal{S}} \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}).$$

Another approach is to show that $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ is a concave functional on the convex set $\mathcal{K}$. Then minimizers can only be attained at the boundary points of the convex set $\mathcal{K}$, i.e., the subset $\mathcal{S}$.

How to solve the χ-subproblem

Linearizing $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ at $\boldsymbol{\chi}^{(k)}$, we obtain

$$
\begin{aligned}
\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}) &\approx \mathcal{E}_\tau(\boldsymbol{\chi}^{(k)}, \boldsymbol{c}^{(k)}) + \sum_{i=1}^{n} \int_\Omega \frac{\delta \mathcal{E}_\tau}{\delta \chi_i}\Big|_{\boldsymbol{\chi}=\boldsymbol{\chi}^{(k)}} \Big(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\Big)\, d\boldsymbol{x} \\
&:= \mathcal{E}_\tau(\boldsymbol{\chi}^{(k)}, \boldsymbol{c}^{(k)}) + \sum_{i=1}^{n} \int_\Omega \varphi_i^{(k)}(\boldsymbol{x}) \Big(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\Big)\, d\boldsymbol{x},
\end{aligned}
$$

where function $\varphi_i^{(k)}$ is given by

$$
0 \le \varphi_i^{(k)}(\boldsymbol{x}) := 2\lambda \sqrt{\frac{\pi}{\tau}} \sum_{j=1, i\neq j}^{n} G_\tau(\boldsymbol{x}) * \chi_j^{(k)}(\boldsymbol{x}) + g_i^{(k)}(\boldsymbol{x}).
$$

How to solve the χ-subproblem (cont'd)

Dropping the constant terms in $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$, then the $\boldsymbol{\chi}$-subproblem becomes

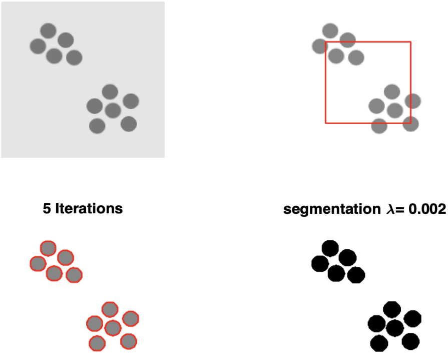
$$\boldsymbol{\chi}^{(k+1)} = \arg\min_{\boldsymbol{\chi}\in\mathcal{K}} \sum_{i=1}^{n} \int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i(\boldsymbol{x})\, d\boldsymbol{x}.$$

Because $\varphi_i^{(k)}(\boldsymbol{x}) \geq 0$ and $\chi_i(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$, the minimizer $\boldsymbol{\chi}^{(k+1)}$ of the above problem can be easily attained at

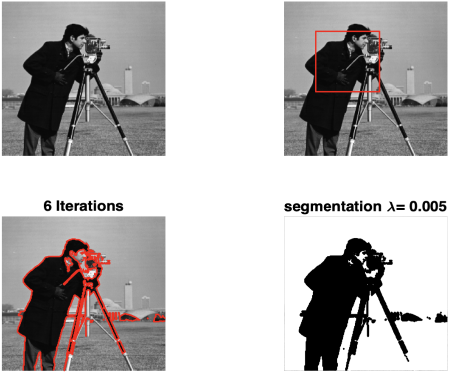
$$\chi_i^{(k+1)}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \varphi_i^{(k)}(\boldsymbol{x}) = \min\limits_{1\leq\ell\leq n} \varphi_\ell^{(k)}(\boldsymbol{x}), \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{for } i = 1, 2, \cdots, n \text{ and } \boldsymbol{x} \in \Omega \setminus \mathcal{C}.$$

Numerical experiment #1



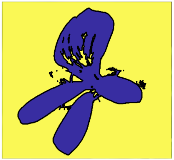
Numerical experiment #2



Numerical experiment #3

17 Iterations

segmentation λ= 0.005



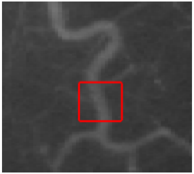
Numerical experiment #4

23 Iterations

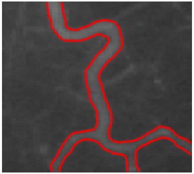
segmentation λ= 0.005

Intensity inhomogeneous images

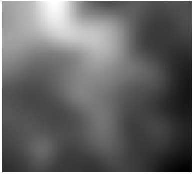
Let $f : \overline{\Omega} \to \mathbb{R}$ be the given grayscale image to be segmented.



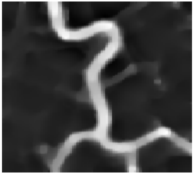
f & initialization C



segmented image



bias field b



corrected image I

Bias field model: $f = bI + n$, where n is the noise

Local intensity clustering model

- C. Li (李純明), R. Huang, Z. Ding, J. C. Gatenby, D. N. Metaxas, and J. C. Gore, A level set method for image segmentation in the presence of intensity inhomogeneities with application to MRI, *IEEE Transactions on Image Processing*, 20 (2011), pp. 2007-2016.
- We need to introduce a bias field model for dealing with intensity inhomogeneous images.
- The level set approach can be replaced by the iterative convolution-thresholding (ICT) scheme.

The bias field model

- The bias field may arise from improper image acquisition in various imaging modalities, especially in the medical imaging domain, such as MRI, PET, CT, etc.
- We assume that the bias field accounts for the intensity inhomogeneity of image. Of course, intensity inhomogeneity can also occur due to spatial variations in illumination.
- The model of bias field in medical images is commonly based upon the assumption that it is *a low-frequency artifact and perceived as a smooth spatially varying function.*
- We assume the multiplicative model with additive noise:

$$f(\boldsymbol{x}) = b(\boldsymbol{x})I(\boldsymbol{x}) + n(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \overline{\Omega},$$

  f is the observed image, I the true image, b the bias field, and n an additive zero-mean Gaussian noise, all are unknown except f.

Local intensity clustering property

- Suppose that the true image I approximately takes n distinct constants $c_1, c_2, \cdots, c_n$ in the disjoint regions $\Omega_1, \Omega_2, \cdots, \Omega_n$.
- Let $\boldsymbol{y} \in \Omega$ and

$$\mathcal{N}(\boldsymbol{y}, \rho) := \{\boldsymbol{x} \in \Omega : \ \|\boldsymbol{x} - \boldsymbol{y}\|_2 < \rho\}.$$

  Then $\{\mathcal{N}(\boldsymbol{y}, \rho) \cap \Omega_i\}_{i=1}^{n}$ forms a natural partition of $\mathcal{N}(\boldsymbol{y}, \rho)$.
- Since b is assumed to be a slowly varying function, it is reasonable that

$$f(\boldsymbol{x}) \approx b(\boldsymbol{y})c_i + n(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathcal{N}(\boldsymbol{y}, \rho) \cap \Omega_i.$$

- The set of these local intensities $\{f(\boldsymbol{x}) | \ \boldsymbol{x} \in \mathcal{N}(\boldsymbol{y}, \rho)\}$ has been naturally *classified into n clusters with the cluster centers $b(\boldsymbol{y})c_i$ in the sense of k-means clustering.*

Local intensity clustering property (cont'd)

- We introduce a nonnegative kernel function $K : \mathbb{R}^2 \to \mathbb{R}$, a truncated Gaussian function,

$$K(\boldsymbol{z}) = \begin{cases} \dfrac{1}{a}\exp\left(-\dfrac{\|\boldsymbol{z}\|_2^2}{2\sigma^2}\right), & \text{for } \|\boldsymbol{z}\|_2 \le \rho, \\ 0, & \text{otherwise,} \end{cases}$$

  $a > 0$ is a normalization constant such that $\int_{\mathbb{R}^2} K(\boldsymbol{z})\, d\boldsymbol{z} = 1$, $\sigma > 0$ is the standard deviation of the Gaussian function.

- We then define a local clustering criterion function $\mathcal{E}(\boldsymbol{y})$ by

$$\mathcal{E}(\boldsymbol{y}) = \sum_{i=1}^{n} \int_{\Omega_i} K(\boldsymbol{y}-\boldsymbol{x})\big(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i\big)^2\, d\boldsymbol{x}.$$

  The smaller the value of $\mathcal{E}(\boldsymbol{y})$, the better the classification of the local intensities $\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{N}(\boldsymbol{y},\rho)\}$.

The local intensity clustering model of Li et al.

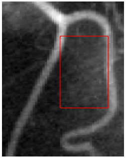
- Li *et al.* defined the optimal partition $\{\Omega_i\}_{i=1}^n$ of Ω as the one such that the local clustering criterion function $\mathcal{E}(\boldsymbol{y})$ is minimized for all $\boldsymbol{y} \in \Omega$.
- They minimized the integral of $\mathcal{E}(\boldsymbol{y})$ with respect to $\boldsymbol{y}$ over Ω, which plays the role of the data fitting term.
- They considered the following local intensity clustering model:

$$\min_{\mathcal{C},b,\boldsymbol{c}} \left(\mu \, |\mathcal{C}| + \int_{\Omega} \sum_{i=1}^{n} \int_{\Omega_i} K(\boldsymbol{y}-\boldsymbol{x}) \big(f(\boldsymbol{x}) - b(\boldsymbol{y}) c_i\big)^2 \, d\boldsymbol{x} \, d\boldsymbol{y} \right),$$

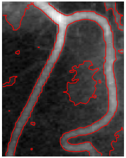
  where $\boldsymbol{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$.
- The energy functional is converted to a level set formulation by representing the disjoint regions $\Omega_1, \Omega_2, \cdots, \Omega_n$ with a number of level set functions.

Numerical experiment: level set formulation

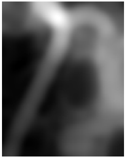
Initial contour



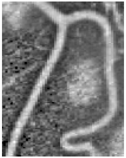
100 iterations



Bias field



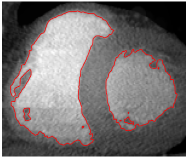
Bias corrected image



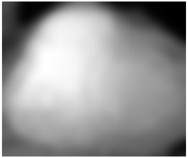
Numerical experiment: level set formulation

Initial contour

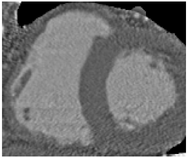
30 iterations



Bias field



Bias corrected image



ICT scheme for solving the model

The ICT scheme can solve the model by considering the following energy functional:

$$\begin{aligned}\mathcal{E}_\tau(\boldsymbol{\chi}, b, \boldsymbol{c}) &= \mu\sqrt{\frac{\pi}{\tau}}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\int_\Omega \chi_i(\boldsymbol{x})\big(G_\tau * \chi_j\big)(\boldsymbol{x})\,d\boldsymbol{x} \\ &\quad + \int_\Omega\sum_{i=1}^{n}\int_\Omega \chi_i(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x})\big(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i\big)^2\,d\boldsymbol{x}\,d\boldsymbol{y}.\end{aligned}$$

We consider the minimization problem:

$$\min_{\boldsymbol{\chi}\in\mathcal{S}, b, \boldsymbol{c}} \mathcal{E}_\tau(\boldsymbol{\chi}, b, \boldsymbol{c}).$$

Three subproblems

Divide the minimization problem into three subproblems: find $\boldsymbol{\chi}^{(k+1)} \in \mathcal{S}$, $b^{(k+1)}$, and $\boldsymbol{c}^{(k+1)}$ sequentially such that

$$\begin{aligned}
\boldsymbol{\chi}^{(k+1)} &= \arg\min_{\boldsymbol{\chi}\in\mathcal{S}} \mathcal{E}_1(\boldsymbol{\chi}), \quad \text{where } \mathcal{E}_1(\boldsymbol{\chi}) := \mathcal{E}_\tau(\boldsymbol{\chi}, b^{(k)}, \boldsymbol{c}^{(k)}), \\
b^{(k+1)} &= \arg\min_{b} \mathcal{E}_2(b), \quad \text{where } \mathcal{E}_2(b) := \mathcal{E}_\tau(\boldsymbol{\chi}^{(k+1)}, b, \boldsymbol{c}^{(k)}), \\
\boldsymbol{c}^{(k+1)} &= \arg\min_{\boldsymbol{c}} \mathcal{E}_3(\boldsymbol{c}), \quad \text{where } \mathcal{E}_3(\boldsymbol{c}) := \mathcal{E}_\tau(\boldsymbol{\chi}^{(k+1)}, b^{(k+1)}, \boldsymbol{c}).
\end{aligned}$$

b-subproblem

We set the functional derivative $\delta\mathcal{E}_2/\delta b$ to be zero,

$$\frac{\delta\mathcal{E}_2}{\delta b} = \sum_{i=1}^{n}\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x})\left(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i^{(k)}\right)\left(-2c_i^{(k)}\right)d\boldsymbol{x} = 0,$$

which implies

$$b(\boldsymbol{y})\int_\Omega\Big(\sum_{i=1}^{n}(c_i^{(k)})^2\chi_i^{(k+1)}(\boldsymbol{x})\Big)K(\boldsymbol{y}-\boldsymbol{x})\,d\boldsymbol{x} = \int_\Omega\Big(\sum_{i=1}^{n}c_i^{(k)}\chi_i^{(k+1)}(\boldsymbol{x})\Big)f(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x}$$

Then

$$b^{(k+1)}(\boldsymbol{y}) = \frac{((J_1 f)*K)(\boldsymbol{y})}{(J_2*K)(\boldsymbol{y})} \quad \text{for } \boldsymbol{y}\in\Omega\setminus\mathcal{C},$$

where

$$J_1(\boldsymbol{x}) = \sum_{i=1}^{n}c_i^{(k)}\chi_i^{(k+1)}(\boldsymbol{x}) \quad \text{and} \quad J_2(\boldsymbol{x}) = \sum_{i=1}^{n}(c_i^{(k)})^2\chi_i^{(k+1)}(\boldsymbol{x}).$$

c-subproblem

Setting all the derivatives of $\mathcal{E}_3$ with respect to c_i to be zero, we obtain

$$\begin{aligned}\frac{\partial \mathcal{E}_3}{\partial c_i} &= 2\int_\Omega \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x})\left(f(\boldsymbol{x}) - b^{(k+1)}(\boldsymbol{y})c_i\right)\left(-b^{(k+1)}(\boldsymbol{y})\right) d\boldsymbol{x}d\boldsymbol{y} \\ &= 0.\end{aligned}$$

Since $K(\boldsymbol{y}-\boldsymbol{x}) = K(\boldsymbol{x}-\boldsymbol{y})$, we can exchange the order of integrations,

$$\begin{aligned}&c_i \int_\Omega \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})\left(b^{(k+1)}(\boldsymbol{y})\right)^2 K(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y}d\boldsymbol{x} \\ &\qquad = \int_\Omega \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})f(\boldsymbol{x})b^{(k+1)}(\boldsymbol{y})K(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y}d\boldsymbol{x}.\end{aligned}$$

It leads to

$$c_i^{(k+1)} = \frac{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})f(\boldsymbol{x})\left(b^{(k+1)} * K\right)(\boldsymbol{x})\, d\boldsymbol{x}}{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})\left((b^{(k+1)})^2 * K\right)(\boldsymbol{x})\, d\boldsymbol{x}} \quad \text{for } i = 1, 2, \cdots, n.$$

χ-subproblem

Linearizing the energy functional $\mathcal{E}_1(\chi)$ at $\chi^{(k)}$, we have

$$\begin{aligned}\mathcal{E}_1(\chi) &\approx \mathcal{E}_1(\chi^{(k)}) + \sum_{i=1}^{n} \int_\Omega \frac{\delta \mathcal{E}_1}{\delta \chi_i}\bigg|_{\chi=\chi^{(k)}} \left(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\right) d\boldsymbol{x} \\ &:= \mathcal{E}_1(\chi^{(k)}) + \sum_{i=1}^{n} \int_\Omega \varphi_i^{(k)}(\boldsymbol{x}) \left(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\right) d\boldsymbol{x}.\end{aligned}$$

where function $\varphi_i^{(k)}$ is given by

$$\begin{aligned}0 \le \varphi_i^{(k)}(\boldsymbol{x}) &:= 2\mu\sqrt{\frac{\pi}{\tau}} \sum_{j=1, j\neq i}^{n} G_\tau(\boldsymbol{x}) * \chi_j^{(k)}(\boldsymbol{x}) \\ &\quad + \int_\Omega K(\boldsymbol{y}-\boldsymbol{x})\left(f(\boldsymbol{x}) - b^{(k)}(\boldsymbol{y}) c_i^{(k)}\right)^2 d\boldsymbol{y}.\end{aligned}$$

χ-subproblem (cont'd)

We then replace the minimization problem with

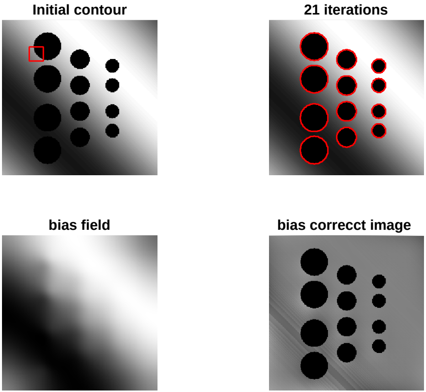
$$\begin{aligned}
\chi^{(k+1)} &= \arg\min_{\chi\in\mathcal{K}}\Big(\mathcal{E}_1(\chi^{(k)}) + \sum_{i=1}^{n}\int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i(\boldsymbol{x})\,d\boldsymbol{x} \\
&\qquad - \sum_{i=1}^{n}\int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i^{(k)}(\boldsymbol{x})\,d\boldsymbol{x}\Big) \\
&= \arg\min_{\chi\in\mathcal{K}}\Big(\sum_{i=1}^{n}\int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i(\boldsymbol{x})\,d\boldsymbol{x}\Big).
\end{aligned}$$

Because $\varphi_i^{(k)}(\boldsymbol{x}) \geq 0$ and $\chi_i(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$, the minimizer $\chi^{(k+1)}$ can be easily attained at

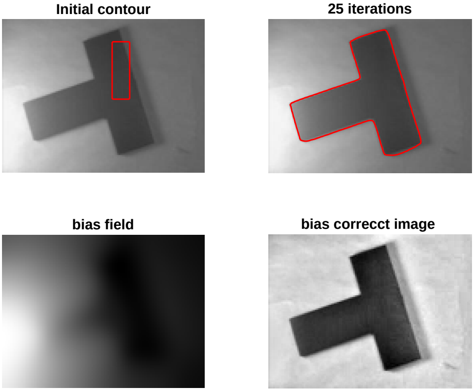
$$\chi_i^{(k+1)}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \varphi_i^{(k)}(\boldsymbol{x}) = \min\limits_{1\leq\ell\leq n} \varphi_\ell^{(k)}(\boldsymbol{x}), \\ 0, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \cdots, n$ and $\boldsymbol{x} \in \Omega \setminus \mathcal{C}$.

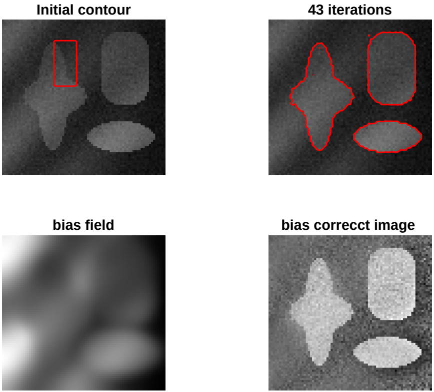
Numerical experiment: ICT scheme



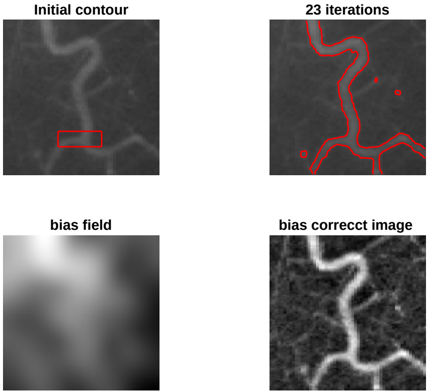
Numerical experiment: ICT scheme



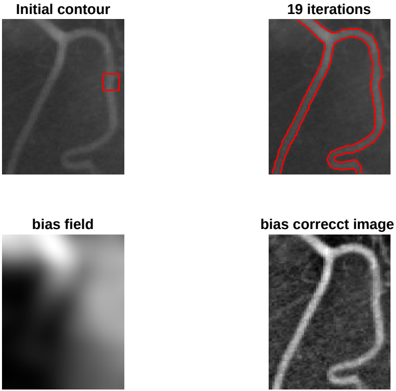
Numerical experiment: ICT scheme



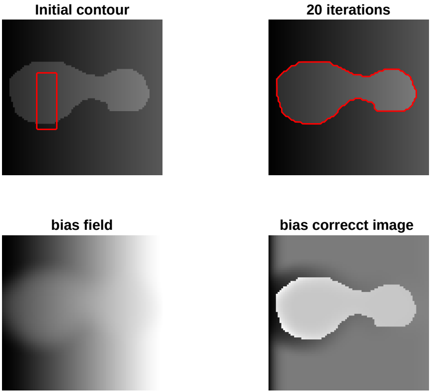
Numerical experiment: ICT scheme



Numerical experiment: ICT scheme



Numerical experiment: ICT scheme



References

1. T. F. Chan and L. A. Vese, Active contours without edges, *IEEE Transactions on Image Processing,* 10 (2001), pp. 266-277.
2. P. Getreuer, Chan-Vese segmentation, *Image Processing On Line,* 2 (2012), pp. 214-224.
3. C. Li, R. Huang, Z. Ding, J. C. Gatenby, D. N. Metaxas, and J. C. Gore, A level set method for image segmentation in the presence of intensity inhomogeneities with application to MRI, *IEEE Transactions on Image Processing,* 20 (2011), pp. 2007-2016.
4. W.-T. Liao, S.-Y. Yang, and C.-S. You, An entropy-weighted local intensity clustering-based model for segmenting intensity inhomogeneous images, *Multimedia Systems,* 30 (2024), article 49.
5. D. Wang, H. Li, X. Wei, X.-P. Wang, An efficient iterative thresholding method for image segmentation, *Journal of Computational Physics,* 350 (2017), pp. 657-667.