MA 3021: Numerical Analysis I
Syllabus and Introduction

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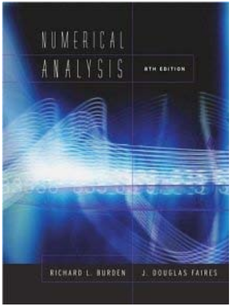
Syllabus

- **Instructor:** Prof. Suh-Yuh Yang (楊肅煜)
  - Office: M315, Hong-Jing Hall
  - Phone: 03-4227151 ext. 65130
- **Office hours**: Tuesday 10:00 ∼ 11:50 am or by appointment.
- **Teaching assistant**: Cheng-Shu You (游承書), M303, Tel: ext. 65138.
- **Prerequisites**: Calculus, Linear Algebra and some knowledge of a high level programming language Fortran/C/C++, or software package MATLAB: `http://matlab.math.ncu.edu.tw/`
- **Assignments**:
  - Approximately every two weeks, will consist of theoretical problems or computer projects.
  - The students are encouraged to discuss homework with other classmates.
  - Direct copying is absolutely not allowed.
- **Examinations:** there will be a midterm and a final (presentation/report).
- **Grading policy:** assignments 40%, midterm 30% and final 30%.

Textbook

Richard L. Burden and J. Douglas Faires, *Numerical Analysis, 8th Edition*, Thomson Brooks/Cole, 2005. (歐亞書局有限公司代理)



Important dates

- The period for adding and dropping a course: 02/03-02/25, 2016.
- The period for withdrawing a course: 03/28-05/13, 2016.
- Spring break: 04/05, 2016 (recess, no class).
- Midterm week: 04/11-04/15, 2016.
- Final week: 06/13-06/17, 2016.

Scientific computing (科學計算) vs. Numerical analysis

- **Problem modelling**
  - physical phenomena: too expensive to perform all tests with prototypes.
  - mathematical model: (differential or integral equations) too complex or very difficult for paper/pencil solution.
  - computational model: (numerical methods) approximation of mathematical model.
- **Scientific computing:** solving mathematical problems numerically on the computer (methods/constructive proofs → algorithms → codes → display).
- **Numerical analysis – mathematics of scientific computing:** it involves the study, development and analysis of algorithms (procedures) for obtaining numerical solutions to various mathematical problems.
- **Scientific computing: interdisciplinary (跨學科)**
  - science/engineering
  - numerical analysis
  - computer science
  - software engineering

This course will cover the following topics

- Mathematical preliminaries
- Solutions of nonlinear equations
- Interpolation and polynomial approximation
- Numerical differentiation and integration
- Direct and iterative methods for solving linear systems
- Numerical ordinary differential equations*
- Numerical partial differential equations*

Topic 1: Mathematical preliminaries

- **Review of calculus.**
- **Taylor's Theorem:** for functions in single or several variables.
- **Rate of convergence:** big O notation.

Topic 2: Solutions of nonlinear equations

- **Question:** given a function $f : \mathbb{R} \to \mathbb{R}$. Find a point $x^* \in \mathbb{R}$ such that

$$f(x^*) = 0.$$

- If $f(x)$ is simple, such as $f(x) = 3x + 1$ or $f(x) = 3x^2 - 4x + 1$, then one can use the root formulas. In general, one has to find the root(s) numerically.
- We will study
  - iterative methods for finding the root (bisection method, secant method, Newton type methods);
  - convergence of the methods;
  - extension to systems of nonlinear equations.

Topic 3: Interpolation and polynomial approximation

- **Polynomial interpolation (多項式插值)**
  We are given $n+1$ data points (x_i, y_i), $i = 0, 1, \cdots, n$, and we seek a polynomial p such that $p(x_i) = y_i$, $0 \le i \le n$, where $y_i = f(x_i)$ for some function f.
- **Hermite interpolation:** the interpolation of a function and some of its derivatives at a set of nodes. e.g., find a polynomial p such that $p(x_i) = f(x_i)$ and $p'(x_i) = f'(x_i)$, $i = 0, 1$.
- **Spline (樣條) interpolation**
  A spline function of degree k is a piecewise polynomial of degree at most k having continuous derivatives of all orders up to $k-1$.

Topic 4: Numerical differentiation and integration

- **Numerical differentiation**
  - Based on Taylor's theorem: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi)$.
  - Based on polynomial interpolation: let p be the Lagrange interpolation of f. Then $f'(x) \approx p'(x)$.
- **Numerical integration based on interpolation:** let p be the Lagrange interpolation of f. Then $\int_a^b f(x)dx \approx \int_a^b p(x)dx$.
- **Gaussian quadrature (高斯積分法):** find A_i and x_i, $i = 0, 1, \cdots, n$, such that $\displaystyle\int_a^b f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)$ and it will be exact for polynomials of degree $\leq 2n+1$.

Topic 5: Direct and iterative methods for solving $Ax = b$

Linear system: find the vector $(x_1, x_2)^\top$ such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

The size of the problem is $n = 2$. For small n, the system can be solved by hand, but for large n (could be as large as $n = 10^6$), one has to use computers. We will study

- vector, matrix, norm
- Gaussian elimination and matrix factorizations
- iterative methods
- error analysis

Topic 6: Numerical ordinary differential equations*

- **Existence and uniqueness theory** of the initial value problem:

$$\begin{cases} x'(t) &= f(t,x), \\ x(t_0) &= x_0. \end{cases}$$

- **Taylor-series method:**

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + \cdots$$

- **Runge-Kutta methods:** in Taylor-series method, we have to determine $x'', x''', x^{(4)}, \cdots$. The Runge-Kutta methods avoid this difficulty.
- **Multistep methods:** e.g., the Adams-Bashforth-formula of order 5:

$$x_{n+1} = x_n + \frac{h}{720}\{1901 f_n - 2774 f_{n-1} + 2616 f_{n-2} - 1274 f_{n-3} + 251 f_{n-4}\}.$$

- **Convergence, stability and consistency:** for multistep method, *convergent* $\Longleftrightarrow$ *stable* + *consistent*.
- **Boundary value problems:** finite difference methods.

Topic 7: Numerical partial differential equations*

- **Parabolic problems:** finite difference method - explicit, implicit.
- **Elliptic problems:** finite difference method, finite element method.
- **Hyperbolic problems:** characteristics.