MA 3021: Numerical Analysis I
Numerical Partial Differential Equations

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What are PDEs?

1. Most physical phenomena in fluid dynamics, heat transfer, electricity, magnetism, or mechanics can be described in general by partial differential equations (PDEs).

2. A PDE is an equation that contains partial derivatives and can be written in the form of
$F(x_1, x_2, \cdots, x_n, u_{x_1}, u_{x_2}, \cdots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \cdots) = 0.$
   - $u(x_1, x_2, \cdots, x_n)$ is a function of n variables $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n$, where u is called the dependent variable and x_i is called the independent variable.
   - $u_{x_i} = \dfrac{\partial u}{\partial x_i}$ is the partial derivative of u in the x_i direction.

3. In general, a PDE may have one solution, many solutions, or no solution at all.

4. Some constrains are often added to the PDE so that the solution is unique. These are often called boundary conditions or initial conditions.

Kinds of PDEs

1. **Linearity:**
   - $F(\cdots) = u_{xx} + xu_{yy}$ is linear.
   - $F(\cdots) = u_{xx} + xu_{yy} + u^2$ is nonlinear.
2. **Order of the PDEs:** The order of the highest derivative that occurs in F is called the order of the PDE. For example,
   - $u_t = u_{xx}$, second order.
   - $u_t = uu_{xxx} + \sin x$, third order.

Second-order linear equations in two variables

Second-order linear equation in two variables takes a general form of

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

- **Parabolic:** parabolic equations describe heat flow and diffusion processes and satisfy $B^2 - 4AC = 0$. For example,
  heat equation: $u_t = u_{xx}$.
- **Hyperbolic:** hyperbolic equations describe vibrating system and wave motion and satisfy $B^2 - 4AC > 0$. For example,
  wave equation: $u_{tt} = u_{xx}$.
- **Elliptic:** elliptic equations describe steady-state phenomena and satisfy $B^2 - 4AC < 0$. For example,
  Poisson's equation: $-(u_{xx} + u_{yy}) = f(x, y)$.

Application of Poisson's equation in heat transfer

Let Ω be an open and bounded domain. Consider

$$-\Delta u := -(u_{xx} + u_{yy}) = f(x, y) \quad \text{on } \Omega$$

is used for describing steady state temperature distribution of some material. Three types of boundary conditions:

- **Dirichlet condition:** $u = g(s)$ on $\partial\Omega$, temperature specified on the boundary.
- **Neumann condition:** $\dfrac{\partial u}{\partial \boldsymbol{n}} = h(s)$ on $\partial\Omega$, heat flow across the boundary (flux) specified, where $\boldsymbol{n}$ is an outward unit normal vector. Note that $\dfrac{\partial u}{\partial \boldsymbol{n}} = \nabla u \cdot \boldsymbol{n}$.
- **Mixed condition:** $\dfrac{\partial u}{\partial \boldsymbol{n}} + \lambda u = g(s)$ on $\partial\Omega$, temperature of the surrounding medium is specified.

1-D heat equation

1. Initial-boundary value problem (IBVP): find $u(x,t)$ such that

$$\begin{cases} u_t = u_{xx} & t > 0, 0 < x < 1, \\ u(x,0) = g(x) & 0 \le x \le 1, \\ u(0,t) = a(t) & t \ge 0, \\ u(1,t) = b(t) & t \ge 0. \end{cases}$$

2. Notations

   $u(x,t)$: unknown temperature in the rod

   x is spatial coordinates and t is time

   $u_{xx} = \frac{\partial^2 u}{\partial x^2}$

   $u_t = \frac{\partial u}{\partial t}$

Finite difference method

- Let
$$\begin{cases} t_j = jk & j \geq 0, \\ x_i = ih & 0 \leq i \leq n+1. \end{cases}$$
Note that $k \neq h$ in general.
- Recall some finite difference approximations:
$$\begin{aligned} f'(x) &\approx \frac{1}{h}\Big(f(x+h) - f(x)\Big) \\ f'(x) &\approx \frac{1}{2h}\Big(f(x+h) - f(x-h)\Big) \\ f''(x) &\approx \frac{1}{h^2}\Big(f(x+h) - 2f(x) + f(x-h)\Big) \end{aligned}$$

Finite difference method - explicit method

1. Let $v \approx u$. Then
$$\frac{1}{h^2}\Big(v(x+h,t) - 2v(x,t) + v(x-h,t)\Big) = \frac{1}{k}\Big(v(x,t+k) - v(x,t)\Big).$$
2. By defining $v_{ij} = v(x_i, t_j)$, we have
$$\frac{1}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big) = \frac{1}{k}\Big(v_{i,j+1} - v_{i,j}\Big).$$
3. Rewrite the above equation to obtain
$$v_{i,j+1} = \frac{k}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big) + v_{i,j}$$
or
$$v_{i,j+1} = \Big(sv_{i-1,j} + (1-2s)v_{i,j} + sv_{i+1,j}\Big),$$
with $s = k/h^2$.

Algorithm

input n , k , M

$h \leftarrow \frac{1}{n+1}$ and $s \leftarrow \frac{k}{h^2}$

$w_i = g(ih)$ $(0 \leq i \leq n+1)$

$t \leftarrow 0$

output $0, t, (w_0, w_1, \cdots, w_{n+1})$

for $j = 1$ **to** M **do**

$v_0 \leftarrow a(jk)$ and $v_{n+1} \leftarrow b(jk)$

  for $i = 1$ **to** n **do**

  $v_i = \left(sw_{i-1} + (1-2s)w_i + sw_{i+1}\right)$

  end do

  $t \leftarrow jk$

  output j , t , $(v_0, v_1, \cdots, v_{n+1})$

  $(w_0, w_1, \cdots, w_{n+1}) \leftarrow (v_0, v_1, \cdots, v_{n+1})$

end do

Stability analysis

1. Assume that $a(t) = b(t) = 0$. At $t_j = jk$, define $V_j = (v_{1,j}, v_{2,j}, \cdots, v_{n,j})^\top$. Then the explicit difference equations becomes $V_{j+1} = AV_j$, where

$$A = \begin{bmatrix} 1-2s & s & & & & \\ s & 1-2s & s & & & \\ & s & 1-2s & s & & \\ & & \ddots & \ddots & \ddots & \\ & & & s & 1-2s & s \\ & & & & s & 1-2s \end{bmatrix}.$$

2. Note that $v_{0,j} = v_{n+1,j} = 0$. We know that exact solution approaches 0 as $t \to \infty$ and therefore the temperature will reduce to zero as $t \to \infty$.

Stability analysis (continued)

1. For the numerical approximation,
$$V_{j+1} = AV_j = A(AV_{j-1}) = \cdots = A^{j+1}V_0.$$
2. Recall the following two statements are equivalent
   - $\lim_{j\to\infty} A^j V = 0$ for all vectors $V \in \mathbb{R}^n$.
   - $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of matrix A.
3. So $s = k/h^2$ should be chosen such that $\rho(A) < 1$.

   The eigenvalues of A are: $\lambda_j = 1 - 2s(1 - \cos\theta_j)$, where $\theta_j = \frac{j\pi}{n+1}, 1 \le j \le n$.

   For $\rho(A) < 1$ we require $-1 < 1 - 2s(1 - \cos\theta_j) < 1$.

   This is true if and only if $s < (1 - \cos\theta_j)^{-1}$.

Stability analysis (continued)

- The worse case $\cos\theta_j = -1$, which does not happen since the largest $\theta_{j=n} = \frac{n\pi}{n+1}$. We have $0 < s < \frac{1}{2}$ or $\frac{k}{h^2} < \frac{1}{2} \Rightarrow k < \frac{h^2}{2}$.
- For example, $h = 0.01 \Rightarrow k < 5 \times 10^{-5} \Rightarrow$ For $0 \le t \le 10$, the number of time step: 0.5×10^6.
- Find eigenvalue of A: Note $A = I - sB$, where

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

If x_i is an eigenvector of B with eigenvalue μ_i then

$$(I - sB)x_i = x_i - s\mu_i x_i = (1 - s\mu_i)x_i = Ax_i.$$

Hence $\lambda_i = 1 - s\mu_i$ is an eigenvalue of A.

Eigenvalues and eigenvectors of the tridiagonal matrix B

Let $x = (\sin\theta, \sin 2\theta, \cdots, \sin n\theta)^\top$. If $\theta = \frac{j\pi}{n+1}$, then x is an eigenvector of B corresponding to the eigenvalue $2 - 2\cos\theta$.

Proof: Please see page 621 in the textbook:

David Kincaid and Ward Cheney, *Numerical Analysis: Mathematics of Scientific Computing, Third Edition*, 2002, Brooks/Cole.

Finite difference method - implicit method

1. We continue to study the initial-boundary value problem: find $u(x,t)$ such that

$$\begin{cases} u_t &= u_{xx} \quad t>0, 0<x<1, \\ u(x,0) &= g(x) \quad 0 \le x \le 1, \\ u(0,t) &= 0 \quad t \ge 0, \\ u(1,t) &= 0 \quad t \ge 0. \end{cases}$$

2. The finite-difference equation :

$$\frac{1}{h^2}\Big(v(x+h,t) - 2v(x,h) + v(x-h,t)\Big) = \frac{1}{k}\Big(v(x,t) - v(x,t-k)\Big).$$

$$\Rightarrow \frac{1}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big) = \frac{1}{k}\Big(v_{i,j} - v_{i,j-1}\Big).$$

3. Let $s = \frac{k}{h^2}$ and rearrange to obtain
$-sv_{i+1,j} + (1+2s)v_{i,j} - sv_{i-1,j} = v_{i,j-1}$, for $1 \le i \le n$.

Stability analysis

1. Let $V_j = (v_{1,j}, v_{2,j}, \cdots, v_{n,j})^\top$ then the method can be written as $AV_j = V_{j-1}$, where A is given by

$$A = \begin{bmatrix} 1+2s & -s & & & \\ -s & 1+2s & -s & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & -s & 1+2s \end{bmatrix}.$$

2. Solve $V_j = A^{-1}V_{j-1} = A^{-1}A^{-1}V_{j-2} \cdots = A^{-j}V_0$.
3. V_0 is known ($u(ih,0)$ initial condition). Here we need $\rho(A^{-1}) < 1$ for stability.

Stability analysis (continued)

1. Since $A = I + sB$, where

$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

   and therefore the eigenvalues of A are given by
   $\lambda_i = 1 + 2s\mu_i = 1 + 2s(1 - \cos\theta_i)$ with $\theta_i = \frac{i\pi}{n+1}$, $1 \le i \le n$.

2. Clearly, $\lambda_i > 1$, since $\lambda_i = 1 + 2s(1 - \cos\theta_i)$

   $\Rightarrow \lambda_i > 1 \Rightarrow \rho(A^{-1}) < 1$.

   $\Rightarrow$ The method is stable for all h and k.

3. Note that we need to solve a tridiagonal system of linear equation to advance each time step.

Algorithm

input n, k, M
$h \leftarrow \frac{1}{n+1}$ and $s \leftarrow \frac{k}{h^2}$
$v_i = g(ih)$ $(1 \le i \le n)$
$t \leftarrow 0$
output $0, t, (v_1, v_2, \cdots, v_n)$
for $i = 1$ **to** $n-1$ **do**
    $c_i = -s$ and $a_i = -s$
end do
for $j = 1$ **to** M **do**
    for $i = 1$ **to** n **do**
        $d_i = 1 + 2s$
    end do
    call tri$(n, a, d, c, v; v)$
    $t \leftarrow jk$
    output $j, t, (v_1, v_2, \cdots, v_n)$
end do

The Crank-Nicolson method

We can combine the previous two methods into a θ-method

$$\frac{\theta}{h^2}\Big(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}\Big) + \frac{1-\theta}{h^2}\Big(v_{i+1,j-1} - 2v_{i,j-1} + v_{i-1,j-1}\Big)$$
$$= \frac{1}{k}\Big(v_{i,j} - v_{i,j-1}\Big).$$

- $\theta = 0 \Longrightarrow$ explicit method
- $\theta = 1 \Longrightarrow$ implicit method
- $\theta = 1/2 \Longrightarrow$ Crank-Nicolson (CN)

The Crank-Nicolson method (continued)

1. Taking $s = \frac{k}{h^2}$ and rewriting the CN method, we obtain
$$-sv_{i-1,j} + (2+2s)v_{i,j} - sv_{i+1,j} = sv_{i-1,j-1} + (2+2s)v_{i,j-1} + sv_{i+1,j-1}.$$

2. Again, let $V_j = (v_{1,j}, v_{2,j}, \cdots, v_{n,j})^\top$ and
$$B = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix}.$$
The method can be written in the matrix form
$$(2I + sB)V_j = (2I - sB)V_{j-1}.$$

Stability analysis

1. For stability, we need $\rho((2I + sB)^{-1}(2I - sB)) < 1$.
2. Set $A = (2I + sB)^{-1}(2I - sB)$ with $V_j = AV_{j-1}$. If x_i is an eigenvector of B then

$$\begin{aligned}(2I - sB)x_i &= 2x_i - sBx_i \\ &= 2x_i - s\mu_i x_i \\ &= (2 - s\mu_i)x_i.\end{aligned}$$

$\Longrightarrow x_i$ is also an eigenvector of A with eigenvalues $\frac{2-s\mu_i}{2+s\mu_i}$.

3. To have $\rho((2 + sB)^{-1}(2 - sB)) < 1$, we get it if $|(2 + s\mu)^{-1}(2 - s\mu)| < 1$.
4. Because $\mu_i = 2(1 - \cos\theta_i)$, we see that $0 < \mu_i < 4$.

   Thus $|\frac{2-s\mu_i}{2+s\mu_i}| < 1, \forall s = \frac{k}{h^2}$.

   So, the CN method is an unconditionally stable method.

Error analysis

1. Recall the explicit method $v_{i,j+1} = s(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}) + v_{i,j}$
Let $u_{i,j}$ be the exact solution at (x_i, t_j). Then the error is given by $e_{i,j} = u_{i,j} - v_{i,j}$.
2. We replace v by $u - e$ to obtain

$$\begin{aligned} u_{i,j+1} - e_{i,j+1} &= s(u_{i-i,j} - 2u_{i,j} + u_{i+1,j}) + u_{i,j} \\ &\quad -s(e_{i-i,j} - 2e_{i,j} + e_{i+1,j}) - e_{i,j} \\ \Longrightarrow e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) \\ &\quad -s(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (u_{i,j+1} - u_{i,j}). \end{aligned}$$

3. Using these formulas

$$\begin{aligned} f''(x) &= \frac{1}{h^2}\big(f(x+h) - 2f(x) + f(x-h)\big) - \frac{h^2}{12} f^{(4)}(\xi), \\ g'(t) &= \frac{1}{k}\big(g(t+k) - g(t)\big) - \frac{k}{2} g''(\tau), \end{aligned}$$

we obtain ($sh^2 = k$ and $u_{xx} = u_t$)

$$\begin{aligned} e_{i,j+1} &= (se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}) - s\Big(h^2 u_{xx}(x_i, t_j) \\ &\quad + \frac{h^4}{12} u_{xxxx}(\xi_i, t_j)\Big) + \Big(ku_t(x_i, t_j) + \frac{k^2}{2} u_{tt}(x_i, \tau_j)\Big). \end{aligned}$$

Error analysis (continued)

$$\Longrightarrow e_{i,j+1} = \left(se_{i-1,j} + (1-2s)e_{i,j} + se_{i+1,j}\right) \\ -kh^2\left(\tfrac{1}{12}u_{xxxx}(\xi_i,t_j) - \tfrac{s}{2}u_{tt}(x_i,\tau_i)\right)$$

1. Let us confine (x,t) to the set $S = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$.
2. Put $M = \dfrac{1}{12}\max|u_{xxxx}(x,t)| + \dfrac{s}{2}\max|u_{tt}(x,t)|$,
   $E_j = (e_{1,j}, e_{2,j}, \cdots, e_{n,j})^\top$, $\|E_j\|_\infty = \max\limits_{1\le i\le n}|e_{ij}|$.
3. We assume $1-2s \ge 0$:
$$\begin{aligned} |e_{i,j+1}| &\le s|e_{i-1,j}| + (1-2s)|e_{ij}| + s|e_{i+1,j}| + kh^2M \\ &\le s\|E_j\|_\infty + (1-2s)\|E_j\|_\infty + s\|E_j\|_\infty + kh^2M \\ &\le \|E_j\|_\infty + kh^2M. \end{aligned}$$
4. Hence,
$$\begin{aligned} \|E_{j+1}\|_\infty &\le \|E_j\|_\infty + kh^2M \le \|E_{j-1}\|_\infty + 2kh^2M \\ &\le \cdots \le \|E_0\|_\infty + (j+1)kh^2M \\ &\Longrightarrow \|E_j\|_\infty \le jkh^2M \Longrightarrow \|E_j\|_\infty \le Th^2M = O(h^2). \end{aligned}$$

Numerical differentiation

Assume that $u \in C^4[a,b]$ and $a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b$ is a uniform partition of $[a,b]$. Then $h_j = h = \frac{b-a}{M+1}$ for $j = 1, 2, \cdots, M+1$.
For $i = 1, 2, \cdots, M$, we have
$u(x_i + h) = u(x_i) + u'(x_i)h + \frac{1}{2}u''(x_i)h^2 + \frac{1}{6}u^{(3)}(x_i)h^3 + \frac{1}{24}u^{(4)}(\xi_{i1})h^4,$
$u(x_i - h) = u(x_i) - u'(x_i)h + \frac{1}{2}u''(x_i)h^2 - \frac{1}{6}u^{(3)}(x_i)h^3 + \frac{1}{24}u^{(4)}(\xi_{i2})h^4,$
for some $\xi_{i1} \in (x_i, x_i + h)$ and $\xi_{i2} \in (x_i - h, x_i)$.

$\therefore u(x_i + h) + u(x_i - h) = 2u(x_i) + u''(x_i)h^2 + \frac{h^4}{24}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$

$\therefore$
$u''(x_i) = \frac{1}{h^2}\{u(x_i + h) - 2u(x_i) + u(x_i - h)\} - \frac{h^2}{24}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$

$\because u \in C^4[a,b]$, $\frac{1}{2}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}$ between $u^{(4)}(\xi_{i1})$ and $u^{(4)}(\xi_{i2})$.

$\therefore$ By IVT, $\exists\, \xi_i$ between ξ_{i1} and ξ_{i2} ($\Rightarrow \xi_i \in (x_i - h, x_i + h)$) such that
$u^{(4)}(\xi_i) = \frac{1}{2}\{u^{(4)}(\xi_{i1}) + u^{(4)}(\xi_{i2})\}.$

$\therefore u''(x_i) = \frac{1}{h^2}\{u(x_i + h) - 2u(x_i) + u(x_i - h)\} - \frac{1}{12}h^2 u^{(4)}(\xi_i),$
for some $\xi_i \in (x_i - h, x_i + h)$. (2nd-order approximation)

Numerical differentiation (continued)

1. **Forward difference:** Assume that $u \in C^2[a,b]$. Then
$u(x_i + h) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(\xi_i)$, for some $\xi_i \in (x_i, x_i + h)$.
$$\therefore u'(x_i) = \frac{1}{h}(u(x_i + h) - u(x_i)) - \frac{h}{2}u''(\xi_i).\text{ (1st-order approx.)}$$
2. **Backward difference:** Assume that $u \in C^2[a,b]$. Then
$u(x_i - h) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(\xi_i)$, for some $\xi_i \in (x_i - h, x_i)$.
$$\therefore u'(x_i) = \frac{1}{h}(u(x_i) - u(x_i - h)) + \frac{h}{2}u''(\xi_i).\text{ (1st-order approx.)}$$
3. **Centered difference:** Assume that $u \in C^3[a,b]$. Then
$u(x_i + h) = u(x_i) + hu'(x_i) + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u^{(3)}(\xi_{i1})$,
$u(x_i - h) = u(x_i) - hu'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u^{(3)}(\xi_{i2})$,
for some $\xi_{i1} \in (x_i, x_i + h)$ and $\xi_{i2} \in (x_i - h, x_i)$.
$$\therefore u'(x_i) = \frac{1}{2h}(u(x_i + h) - u(x_i - h)) + \frac{h^2}{6}u''(\xi_i).\text{ (2nd-order approximation)}$$

FDM for a two-point boundary value problem

1. Consider the 1-D two-point BVP:

$$\begin{cases} -u''(x) &= f(x) \quad x \in (0,1), \\ u(0) &= u(1) = 0. \end{cases}$$

2. The interval $[0,1]$ is discretized uniformly by taking the $n+2$ points, $x_i = ih$, for $i = 0, 1, \cdots, n+1$, where $h = 1/(n+1)$.
3. Let $v_i \approx u(x_i)$, $i = 1, 2, \cdots, n$, and $v_0 := u(x_0) = 0$, $v_{n+1} := u(x_{n+1}) = 0$ are known due to the Dirichlet BC.
4. If the centered difference approximation is used for u'', the above equation can be expressed as

$$-\left(\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}\right) = f_i, \quad i = 1, 2, \cdots, n,$$

where $f_i := f(x_i)$.

The resulting linear system

The linear system obtained is of the form

$$AV = F,$$

where

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

$$V = (v_1, v_2, \cdots, v_n)^\top \quad \text{and} \quad F = (h^2 f_1, h^2 f_2, \cdots, h^2 f_n)^\top.$$

Eigen properties of A

1. The matrix A has n eigenvalues, and since A is symmetric, all eigenvalues must be real.
2. Note that the eigenvalues of A are given by
$$\lambda_j = 2 - 2\cos(j\theta) > 0, j = 1, 2, \cdots, n,$$
and the eigenvector associated with each λ_j is given by
$$V_j = (\sin(j\theta), \sin(2j\theta), \cdots, \sin(nj\theta))^\top,$$
where $\theta = \frac{\pi}{n+1}$.
3. $\lambda_{\max} = 2 - 2\cos(\frac{n\pi}{n+1})$ and $\lambda_{\min} = 2 - 2\cos(\frac{\pi}{n+1})$.
4. What is the condition number of A?
$$\kappa(A) = \frac{\sin^2 \frac{n\pi}{2(n+1)}}{\sin^2 \frac{\pi}{2(n+1)}} \approx \frac{1}{(\frac{\pi}{2(n+1)})^2} \approx O\left(n^2\right) \approx O\left(\frac{1}{h^2}\right).$$

FDM for a 2-D boundary value problem

1. Consider Poisson's problem,
$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = f & \text{in } \Omega := (0,1)\times(0,1), \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2. Define the mesh size $h = \frac{1}{n+1}$, the collection of mesh points $(x_{1i}, x_{2j}) = (ih, jh)$, the approximate solution at the mesh points $v_{ij} \approx u(x_{1j}, x_{2j})$, $i, j = 0, 1, \cdots, n+1$.

   There are n^2 interior points $\approx \frac{1}{h^2}$. (in 3D, $\approx \frac{1}{h^3}$ number of points).

3. The FD equations
$$\begin{cases} \dfrac{v_{i-1j} - 2v_{ij} + v_{i+1j}}{h^2} + \dfrac{v_{ij-1} - 2v_{ij} + v_{ij+1}}{h^2} = f_{ij}, \\ v_{0j} = v_{n+1j} = v_{i0} = v_{in+1} = 0. \end{cases}$$

For example $n = 3$: natural ordering

1. We order the unknown quantities in the natural ordering
$$V = (v_{11}, v_{21}, v_{31}, v_{12}, v_{22}, v_{n2}, v_{13}, v_{23}, v_{33})^\top.$$

2. Then the corresponding linear system can be written as (see Text, page 631)
$$AV = \begin{bmatrix} B & -I & \\ -I & B & -I \\ & -I & B \end{bmatrix} V = F \quad \text{with} \quad B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$$

3. block-tridiag matrix; symmetric $a_{ij} = a_{ji}$; sparse, number of nonzeros per row ≈ 5 (independent of the mesh size h) number of nonzeros $\approx 5n$ (linear in n).

References

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