MA 7007: Numerical Solution of Differential Equations I
Finite Difference Approximations

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Introduction

Assumption: function $u(x)$ is sufficiently smooth, i.e., we can differentiate $u(x)$ several times and each derivative is well-defined bounded function over an interval containing a particular point of interest $\bar{x}$.

Suppose we want to approximate $u'(\bar{x})$ by a finite difference approximation. Let h be a small value ($h > 0$).

1. $u'(\bar{x}) \approx D_+u(\bar{x}) := \dfrac{u(\bar{x}+h) - u(\bar{x})}{h}$ (first order accurate approximation)
2. $u'(\bar{x}) \approx D_-u(\bar{x}) := \dfrac{u(\bar{x}) - u(\bar{x}-h)}{h}$ (first order accurate approximation)
3. $u'(\bar{x}) \approx D_0u(\bar{x}) := \dfrac{u(\bar{x}+h) - u(\bar{x}-h)}{2h} = \dfrac{1}{2}\left\{D_+u(\bar{x}) + D_-u(\bar{x})\right\}$ (second order accurate approximation)
4. $u'(\bar{x}) \approx D_3u(\bar{x}) := \dfrac{1}{6h}\left\{2u(\bar{x}+h) + 3u(\bar{x}) - 6u(\bar{x}-h) + u(\bar{x}-2h)\right\}$ (third order accurate approximation)

Various finite difference approximations to $u'(\bar{x})$

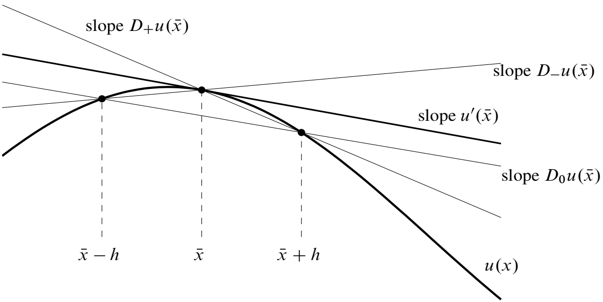

Figure 1.1. *Various approximations to $u'(\bar{x})$ interpreted as the slope of secant lines.*

Example

$u(x) = \sin(x), \bar{x} = 1, u'(1) = \cos(1) = 0.5403023$

Table 1.1. *Errors in various finite difference approximations to* $u'(\bar{x})$.

h	$D_+u(\bar{x})$	$D_-u(\bar{x})$	$D_0u(\bar{x})$	$D_3u(\bar{x})$
1.0e−01	−4.2939e−02	4.1138e−02	−9.0005e−04	6.8207e−05
5.0e−02	−2.1257e−02	2.0807e−02	−2.2510e−04	8.6491e−06
1.0e−02	−4.2163e−03	4.1983e−03	−9.0050e−06	6.9941e−08
5.0e−03	−2.1059e−03	2.1014e−03	−2.2513e−06	8.7540e−09
1.0e−03	−4.2083e−04	4.2065e−04	−9.0050e−08	6.9979e−11

We see that

$$
\begin{aligned}
D_+u(\bar{x}) - u'(\bar{x}) &\approx -0.42h, \\
D_0u(\bar{x}) - u'(\bar{x}) &\approx -0.09h^2, \\
D_3u(\bar{x}) - u'(\bar{x}) &\approx 0.007h^3,
\end{aligned}
$$

Example (continued)

If the error $E(h)$ behaves like $E(h) \approx Ch^p$, then $\log|E(h)| \approx \log|C| + p\log h$.

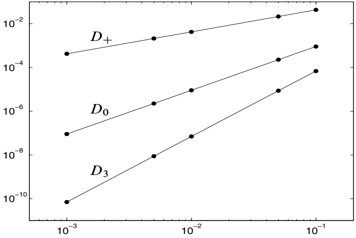

Figure 1.2. *The errors in* $Du(\bar{x})$ *from Table* 1.1 *plotted against h on a log-log scale.*

For $0 < h_2 < h_1$ sufficiently small, we expect $E(h_1) \approx Ch_1^p$ and $E(h_2) \approx Ch_2^p$. Then

$$\log\frac{|E(h_1)|}{|E(h_2)|} \approx \log\frac{|C|h_1^p}{|C|h_2^p} = \log\left(\frac{h_1}{h_2}\right)^p = p\log\frac{h_1}{h_2}.$$

Therefore, the order of convergence can be estimated by

$$p \approx \frac{\log(|E(h_1)|/|E(h_2)|)}{\log(h_1/h_2)}.$$

Taylor's Theorem and "O" notation

1. If $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on (a,b), then for any points c and x in $[a,b]$ we have

$$f(x) = P_n(x) + E_n(x),$$

where the n-th Taylor polynomial $P_n(x)$ is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k$$

and the remainder (error) term $E_n(x)$ is given by

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1}$$

for some point ξ between c and x (means that either $c < \xi < x$ or $x < \xi < c$).

2. If $f(h)$ and $g(h)$ are two functions of h, then we say that $f(h) = O(g(h))$ as $h \to 0$ if $|f(h)| \leq C|g(h)|$ fo all h sufficiently small.

Truncation errors of $D_+u(\bar{x})$ and $D_-u(\bar{x})$

1. Assume that $u \in C^2[a,b]$, $\bar{x} \in (a,b)$, and $0 < h \ll 1$. By Taylor's Theorem,
$$u(\bar{x}+h) = u(\bar{x}) + hu'(\bar{x}) + \frac{1}{2}h^2u''(\xi),$$
for some $\xi \in (\bar{x}, \bar{x}+h)$. Therefore, we have
$$u'(\bar{x}) = \underbrace{\frac{u(\bar{x}+h) - u(\bar{x})}{h}}_{D_+u(\bar{x})} - \frac{1}{2}hu''(\xi) = D_+u(\bar{x}) - \frac{1}{2}hu''(\xi).$$
$$\because\ |\underbrace{u'(\bar{x}) - D_+u(\bar{x})}_{f(h)}| = |-\tfrac{1}{2}hu''(\xi)| \le C|\underbrace{h}_{g(h)}| = Ch$$
$$\therefore\ u'(\bar{x}) - D_+u(\bar{x}) = O(h) \quad \Longrightarrow \quad u'(\bar{x}) = D_+u(\bar{x}) + O(h)$$
2. Assume that $u \in C^2[a,b]$, $\bar{x} \in (a,b)$, and $0 < h \ll 1$. By Taylor's Theorem,
$$u(\bar{x}-h) = u(\bar{x}) - hu'(\bar{x}) + \frac{1}{2}h^2u''(\xi),$$
for some $\xi \in (\bar{x}-h, \bar{x})$. Therefore, we have
$$u'(\bar{x}) = \underbrace{\frac{u(\bar{x}) - u(\bar{x}-h)}{h}}_{D_-u(\bar{x})} + \frac{1}{2}hu''(\xi) = D_-u(\bar{x}) + \frac{1}{2}hu''(\xi) = D_-u(\bar{x}) + O(h).$$

Truncation error of $D_0 u(\bar{x})$

Assume that $u \in C^3[a,b]$, $\bar{x} \in (a,b)$, and $0 < h \ll 1$. By Taylor expansion,

$$\begin{aligned} u(\bar{x}+h) &= u(\bar{x}) + hu'(\bar{x}) + \frac{1}{2}h^2u''(\bar{x}) + \frac{1}{6}h^3u'''(\xi_1), \\ u(\bar{x}-h) &= u(\bar{x}) - hu'(\bar{x}) + \frac{1}{2}h^2u''(\bar{x}) - \frac{1}{6}h^3u'''(\xi_2), \end{aligned}$$

for some $\xi_1 \in (\bar{x}, \bar{x}+h)$ and $\xi_2 \in (\bar{x}-h, \bar{x})$. Therefore, we have

$$\begin{aligned} u'(\bar{x}) &= \overbrace{\frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}}^{D_0u(\bar{x})} - \frac{1}{2h}\frac{h^3}{6}\Big(u'''(\xi_1) + u'''(\xi_2)\Big) \\ &= D_0u(\bar{x}) - \frac{h^2}{6}\frac{u'''(\xi_1) + u'''(\xi_2)}{2}. \end{aligned}$$

Since $\frac{u'''(\xi_1)+u'''(\xi_2)}{2}$ is between $u'''(\xi_1)$ and $u'''(\xi_2)$ and $u \in C^3[a,b]$, by the Intermediate Value Theorem, there exists a ξ between ξ_1 and ξ_2, hence $\xi \in (\bar{x}-h, \bar{x}+h)$, such that

$$u'''(\xi) = \frac{u'''(\xi_1) + u'''(\xi_2)}{2}.$$

Therefore, we have

$$u'(\bar{x}) = D_0u(\bar{x}) - \frac{h^2}{6}u'''(\xi) = D_0u(\bar{x}) + O(h^2).$$

Truncation error of $D_3 u(\bar{x})$

Assume that $u \in C^4[a,b]$, $\bar{x} \in (a,b)$, and $0 < h \ll 1$. By Taylor expansion,

$$\begin{aligned}
u(\bar{x}+h) &= u(\bar{x}) + hu'(\bar{x}) + \frac{1}{2}h^2u''(\bar{x}) + \frac{1}{6}h^3u'''(\bar{x}) + \frac{1}{4!}h^4u^{(4)}(\xi_1), \\
u(\bar{x}-h) &= u(\bar{x}) - hu'(\bar{x}) + \frac{1}{2}h^2u''(\bar{x}) - \frac{1}{6}h^3u'''(\bar{x}) + \frac{1}{4!}h^4u^{(4)}(\xi_2), \\
u(\bar{x}-2h) &= u(\bar{x}) - 2hu'(\bar{x}) + \frac{1}{2}(2h)^2u''(\bar{x}) - \frac{1}{6}(2h)^3u'''(\bar{x}) + \frac{1}{4!}(2h)^4u^{(4)}(\xi_3),
\end{aligned}$$

for some $\xi_1 \in (\bar{x}, \bar{x}+h)$, $\xi_2 \in (\bar{x}-h, \bar{x})$ and $\xi_3 \in (\bar{x}-2h, \bar{x})$. With these identities, we can verify that

$$\begin{aligned}
u'(\bar{x}) &= \overbrace{\frac{1}{6h}\left\{2u(\bar{x}+h) + 3u(\bar{x}) - 6u(\bar{x}-h) + u(\bar{x}-2h)\right\}}^{D_3u(\bar{x})} \\
&\qquad - \frac{1}{6h}\left\{\frac{2}{4!}h^4u^{(4)}(\xi_1) - \frac{6}{4!}h^4u^{(4)}(\xi_2) + \frac{16}{4!}h^4u^{(4)}(\xi_3)\right\} \\
&= D_3u(\bar{x}) + O(h^3).
\end{aligned}$$

Method of undetermined coefficients

Suppose we want to derive a finite difference approximation to $u'(\bar{x})$,

$$u'(\bar{x}) \approx D_2 u(\bar{x}) := au(\bar{x}) + bu(\bar{x}-h) + cu(\bar{x}-2h),$$

where the coefficients a, b, c need to be determined. Using the Taylor expansion,

$$D_2 u(\bar{x}) = (a+b+c)u(\bar{x}) - (b+2c)hu'(\bar{x}) + \frac{1}{2}(b+4c)h^2 u''(\bar{x}) - \frac{1}{6}(b+8c)h^3 u'''(\bar{x}) + O(h^4).$$

Since $u'(\bar{x}) \approx D_2 u(\bar{x})$, we need

$$a+b+c=0, \quad b+2c=-1/h, \quad b+4c=0.$$

Therefore $a = 3/(2h), b = -2/h, c = 1/(2h)$, and we have

$$D_2 u(\bar{x}) = \frac{1}{2h}\Big\{3u(\bar{x}) - 4u(\bar{x}-h) + u(\bar{x}-2h)\Big\}$$

and

$$D_2 u(\bar{x}) - u'(\bar{x}) = -\frac{1}{6}(b+8c)h^3 u'''(\bar{x}) + O(h^4) = -\frac{1}{3}h^2 u'''(\bar{x}) + O(h^3) = O(h^2).$$