MA 7007: Numerical Solution of Differential Equations I
Advection Equations and Hyperbolic Systems

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Advection equation (平流方程)

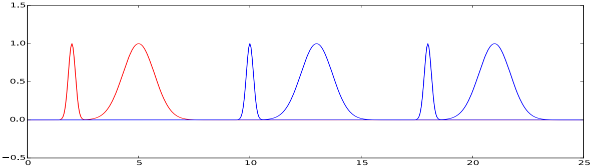
We consider the scalar advection equation

$$u_t + au_x = 0, \quad \text{for } -\infty < x < \infty, t > 0,$$

where a is a constant. For the Cauchy problem we also need initial data

$$u(x,0) = \eta(x).$$

- This is the simplest example of a *hyperbolic* equation.
- The exact solution is given by $u(x,t) = \eta(x - at)$ and a is the velocity of the wave profile. Note that let $x - at = c$ then $t = (x - c)/a$.



Forward difference method and Lax-Friedrichs method

One natural discretization of $u_t + au_x = 0$ would be the *forward difference method:*

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{a}{2h}\left(U_{j+1}^n - U_{j-1}^n\right),$$

where we use the standard centered difference in space and a forward difference in time. This is an explicit method since we can compute each U_j^{n+1} explicitly in terms of the previous data:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right).$$

In practice this method is not useful because of stability considerations, as we will see later. A minor modification gives a more useful method:

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right),$$

which we call the *Lax-Friedrichs method*.

Some remarks on the Lax-Friedrichs method

- Because of the low accuracy, this method is not commonly used in practice!
- We will show later that the Lax-Friedrichs method is Lax-Richtmyer stable, provided $\left|\frac{ak}{h}\right| \leq 1$. This stability restriction allows us to use a time step $k = O(h)$.
- Note that

$$\begin{aligned} u_x(x,t) &= \eta'(x-at), \\ u_t(x,t) &= -au_x(x,t) = -a\eta'(x-at). \end{aligned}$$

The time derivative u_t is larger in magnitude than u_x by a factor of a, and so we would expect the time step required to achieve temporal resolution consistent with the spatial resolution h to be smaller by a factor of a. This suggests that the relation $k \approx h/a$ would be reasonable in practice.

Initial boundary value problem (IBVP)

- **Cauchy problem:** advection equation on infinite 1D domain

$$u_t + au_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

  with initial data $u(x,0) = \eta(x)$ for $-\infty < x < \infty$.
- **Initial boundary value problem:** advection equation on finite 1D domain

$$u_t + au_x = 0, \quad 0 < x < 1, \quad t > 0$$

  with initial data $u(x,0) = \eta(x)$, $0 \le x \le 1$ and boundary condition at the inflow boundary:
  - If $a > 0$, need a boundary condition at $x = 0$: $u(0,t) = g_0(t)$ for $t \ge 0$. In this case, $x = 0$ is called the inflow boundary and $x = 1$ is called the outflow boundary.
  - If $a < 0$, need a boundary condition at $x = 1$: $u(1,t) = g_0(t)$ for $t \ge 0$. In this case, $x = 1$ is called the inflow boundary and $x = 0$ is called the outflow boundary.

Periodic boundary conditions and MOL discretization

- For analysis purposes we can obtain a nice MOL discretization if we consider the *periodic boundary conditions*:

$$u(0,t) = u(1,t), \quad t \geq 0,$$

  and in this case, the value $U_0(t) = U_{m+1}(t)$ along the boundaries is another unknown and we must introduce one of these into the vector $U(t)$.

- If we introduce $U_{m+1}(t)$, then we have the vector of grid values $U(t) = [U_1(t), U_2(t), \cdots, U_{m+1}(t)]^\top$. For $2 \leq j \leq m$ we have the ODE

$$U_j'(t) = -\frac{a}{2h}\Big(U_{j+1}(t) - U_{j-1}(t)\Big),$$

  while the first and last equations are modified using the periodicity:

$$\begin{aligned} U_1'(t) &= -\frac{a}{2h}\Big(U_2(t) - U_{m+1}(t)\Big), \\ U_{m+1}'(t) &= -\frac{a}{2h}\Big(U_1(t) - U_m(t)\Big). \end{aligned}$$

Stability analysis

The IVP of the system of ODEs can be written as:

$$U'(t) = AU(t) \qquad \text{and} \qquad U_j(0) = \eta(x_j) \text{ for } 1 \le j \le m+1,$$

where

$$A = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ 1 & & & & -1 & 0 \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+1)}.$$

Note that this matrix is skew-symmetric ($A^\top = -A$) and so its eigenvalues must be pure imaginary. In face, the eigenvalues are

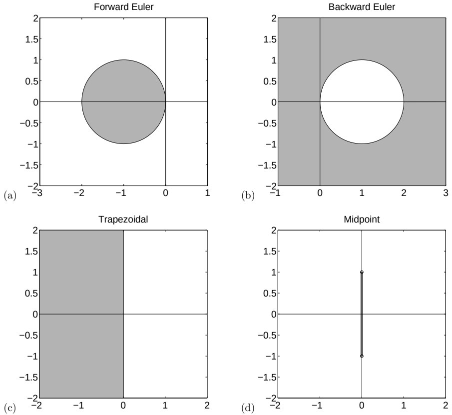
$$\lambda_p = -\frac{ia}{h}\sin(2\pi p h), \quad p = 1, 2, \cdots, m+1.$$

The corresponding eigenvector u^p has components

$$u_j^p = e^{2\pi i p j h}, \quad j = 1, 2, \cdots, m+1.$$

The eigenvalues lie on the imaginary axis between $-ia/h$ and ia/h.

Recall stability region of some methods for IVP



The forward difference method

Applying the forward Euler time discretization to the IVP, $U'(t) = AU(t)$ and $U_j(0) = \eta(x_j)$ for $1 \le j \le m+1$, we have the forward difference method:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right).$$

- **Stability:** We must require $|1 + k\lambda| \le 1$ for each eigenvalue and the stability region $\mathcal{S}$ is the unit circle centered at -1. However, this mehtod is *unstable* for any fixed mesh ratio k/h since the eigenvalues λ_p are imaginary, the values $k\lambda_p$ will not lie in $\mathcal{S}$.
- **Convergence:** This method will be convergent if we let $k \to 0$ *faster than h*. Suppose we take $k = h^2$, $B = I + kA$. Then

$$\|1 + kA\|_2^2 = \rho\Big((I - kA)(I + kA)\Big) = \rho\left(I - k^2A^2\right) \le 1 + k^2\frac{a^2}{h^2} = 1 + a^2k.$$

  Thus, if $nk \le T$, then we have

$$\|(I + kA)^n\|_2 \le (1 + a^2k)^{n/2} \le e^{a^2T/2}.$$

  It is Lax-Richtmyer stable and hence the method is convergent if $k = h^2$.

The leapfrog method

If we apply the midpoint method to the IVP, $U'(t) = AU(t)$ and $U_j(0) = \eta(x_j)$ for $1 \leq j \leq m+1$, i.e.,

$$U^{n+1} = U^{n-1} + 2kAU^n,$$

then we obtain the so-called *leapfrog method* for the advection equation,

$$U_j^{n+1} = U_j^{n-1} - \frac{ak}{h}(U_{j+1}^n - U_{j-1}^n).$$

This is a 3-level explicit method and is second order accurate in both space and time.

Stability: Rcall from Section 7.3 (see also page 8) that the stability region of the midpoint method is the interval $i\alpha$ for $-1 < \alpha < 1$ of the imaginary axis. Hence, the leapfrog method is stable for the advection equation, provided $|ak/h| < 1$ is satisfied ($\Longrightarrow k\lambda_p \in \mathcal{S}_{\text{midpoint}}$).

The Lax-Friedrichs method

- Recall the Lax-Friedrichs method

$$U_j^{n+1} = \frac{1}{2}\left(U_{j-1}^n + U_{j+1}^n\right) - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right).$$

We can rewrite the method as (by "$+U_j^n - U_j^n$")

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{1}{2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right).$$

This can be rearranged to give

$$\frac{U_j^{n+1} - U_j^n}{k} + a\left(\frac{U_{j+1}^n - U_{j-1}^n}{2h}\right) = \frac{h^2}{2k}\left(\frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2}\right).$$

- The right-hand side vanishes as $k, h \to 0$ (assuming k/h is fixed) $\Longrightarrow$ consistent with $u_t + au_x = 0$.
- It looks more like a discretization of the advection-diffusion equation,

$$u_t + au_x = \varepsilon u_{xx}, \quad \text{where } \varepsilon = h^2/(2k).$$

Stability analysis of the Lax-Friedrichs method

- The Lax-Friedrichs method can be viewed as a froward Euler discretization of the system of ODEs $U'(t) = A_\varepsilon U(t)$ with $\varepsilon = h^2/(2k)$ and

$$A_\varepsilon = -\frac{a}{2h}\begin{bmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ 1 & & & & -1 & 0 \end{bmatrix} + \frac{\varepsilon}{h^2}\begin{bmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{bmatrix}.$$

- The eigenvalues of A_ε are

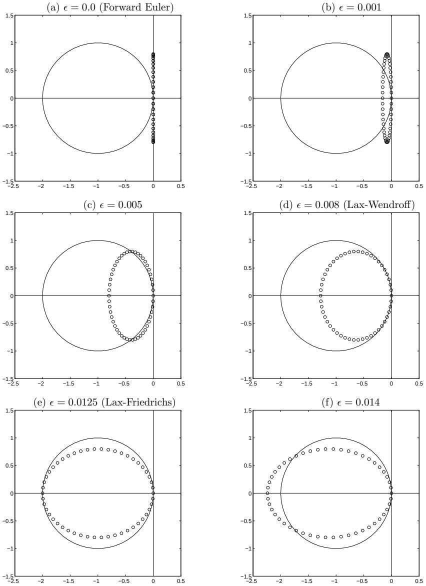
$$\mu_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\varepsilon}{h^2}\Big(1-\cos(2\pi ph)\Big), \quad p = 1, 2, \cdots, m+1.$$

- The values $k\mu_p$ lie on an ellipse centered at

$$-2k\varepsilon/h^2 = -2k(h^2/2k)/h^2 = -1$$

with semi-axes of length $2k\varepsilon/h^2 = 1$ in the x-direction and ak/h in the y-direction. Thus, the Lax-Friedrichs method is stable if $|ak/h| \leq 1$ ($\Longrightarrow k\mu_p \in \mathcal{S}_{\text{forward Euler}}$).

$k\mu_p$ for various values ε: $h = 1/50$, $k = 0.8h$ and $a = 1$



The Lax-Wendroff method

Applying the Taylor series expansion directly to $u_t + au_x = 0$, we have

$$u(x, t+k) = u(x,t) + ku_t(x,t) + \frac{1}{2}k^2 u_{tt}(x,t) + \cdots$$

Replacing u_t by $-au_x$ and $u_{tt} = (-au_x)_t = -a(u_t)_x = -a(-au_x)_x = a^2 u_{xx}$ gives

$$u(x, t+k) = u(x,t) - kau_x(x,t) + \frac{1}{2}k^2 a^2 u_{xx}(x,t) + \cdots$$

If we now use the standard centered approximation to u_x and u_{xx} and drop the higher order terms, we obtain the *Lax-Wendroff method*

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{a^2k^2}{2h^2}\left(U_{j-1}^n - 2U_j^n + U_{j+1}^n\right).$$

This is a 2-level, 3-point, explicit method and is second order accurate in both space and time.

Stability analysis of the Lax-Wendroff method

- The Lax-Wendroff method can be viewed as forward Euler time discretization applied to $U'(t) = A_\varepsilon U(t)$ with $\varepsilon = a^2k/2$ (instead of the value $\varepsilon = h^2/(2k)$ used in Lax-Friedrichs). Then we have

$$k\mu_p = -i\left(\frac{ak}{h}\right)\sin(p\pi h) - \left(\frac{ak}{h}\right)^2\Big(1 - \cos(p\pi h)\Big).$$

- These values all lie on an ellipse centered at $-(ak/h)^2$ with semi-axes of length $(ak/h)^2$ and $|ak/h|$.
- The method is stable if $|ak/h| \le 1$ ($\Longrightarrow k\mu_p \in \mathcal{S}_{\text{forward Euler}}$), with exactly the same time step restriction as required for Lax-Friedrichs.
- The Lax-Wendroff method has the minimal amount of numerical damping needed to bring the values $k\mu_p$ within the stability region, see the figure on page 13.

Upwind methods

- Consider one-sided approximation to u_x in the advection equation, e.g.,

$$u_x(x_j,t) \approx \frac{1}{h}\left(U_j - U_{j-1}\right) \quad \text{or} \quad u_x(x_j,t) \approx \frac{1}{h}\left(U_{j+1} - U_j\right).$$

- For $a > 0$:

$$U_j^{n+1} = U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right), \quad \text{stable if } 0 < \frac{ak}{h} \le 1.$$

  For $a < 0$:

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_{j+1}^n - U_j^n), \quad \text{stable if } -1 \le \frac{ak}{h} < 0.$$

  First order accurate in both space and time.
- It is natural to use nonsymmetric approximation to u_x in the advection equation, since the equation models translation at speed a.

Stability analysis of upwind methods

- The upwind method for $a > 0$ can be rewritten as

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{ak}{2h}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right),$$

  which is the froward Euler discretization of $U'(t) = A_\varepsilon U(t)$ with $\varepsilon = ah/2$. Then

$$k\mu_p = -\frac{iak}{h}\sin(2\pi ph) - \frac{2\varepsilon k}{h^2}\left(1 - \cos(2\pi ph)\right), \quad p = 1, 2, \cdots, m+1.$$

  These values all lie on a circle centered at $-ak/h$ with radius ak/h. The method is stable, provided $0 < ak/h \leq 1$. (stable if $-1 \leq ak/h < 0$ for $a < 0$)

- The three methods Lax-Wendroff, upwind, and Lax-Friedrichs, can all be written as approximations to the advection-diffusion equation $u_t + au_x = \varepsilon u_{xx}$ with different ε,

$$\varepsilon_{\mathrm{LW}} = \frac{a^2k}{2} = \frac{ah\nu}{2}, \quad \varepsilon_{\mathrm{UP}} = \frac{ah}{2}, \quad \varepsilon_{\mathrm{LF}} = \frac{h^2}{2k} = \frac{ah}{2\nu}.$$

  where $\nu = \frac{ak}{h}$. Note that $\varepsilon_{\mathrm{LW}} = \nu\varepsilon_{\mathrm{UP}}$ and $\varepsilon_{\mathrm{UP}} = \nu\varepsilon_{\mathrm{LF}}$. If $0 < \nu < 1$ then $\varepsilon_{\mathrm{LW}} < \varepsilon_{\mathrm{UP}} < \varepsilon_{\mathrm{LF}}$ and the method is stable for any value of ε between $\varepsilon_{\mathrm{LW}}$ and $\varepsilon_{\mathrm{LF}}$.

The Beam-Warming method

A second order accurate method with the same one-sided character can be derived by following the derivation of the Lax-Wendroff method, but using one-sided approximations to u_x and u_{xx} at x_j. This results in the *Beam-Warming method:*

- For $a > 0$,

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right) + \frac{a^2k^2}{2h^2}\left(U_j^n - 2U_{j-1}^n + U_{j-2}^n\right).$$

  $\Longrightarrow$ stable if $0 < \nu \leq 2$.

- For $a < 0$,

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(-3U_j^n + 4U_{j+1}^n - U_{j+2}^n\right) + \frac{a^2k^2}{2h^2}\left(U_j^n - 2U_{j+1}^n + U_{j+2}^n\right).$$

  $\Longrightarrow$ stable if $-2 \leq \nu < 0$.

Von Neumann analysis

Let $\nu = ak/h$, $U_j^n := e^{ijh\xi}$ and $U_j^{n+1} = g(\xi)U_j^n$.

- **Upwind method:** For $a > 0$, we have

$$g(\xi) = (1-\nu) + \nu e^{-i\xi h},$$

and

$$|g(\xi)| \le 1 \Longleftrightarrow 0 < \nu \le 1.$$

- **Lax-Friedrichs:**

$$\begin{aligned} g(\xi) &= \frac{1}{2}\left(e^{-i\xi h} + e^{i\xi h}\right) - \frac{1}{2}\nu\left(e^{i\xi h} - e^{-i\xi h}\right) \\ &= \cos(\xi h) - \nu i \sin(\xi h). \end{aligned}$$

We have

$$|g(\xi)|^2 = \cos^2(\xi h) + \nu^2 \sin^2(\xi h) \le 1 \Longleftrightarrow |\nu| \le 1.$$

Von Neumann analysis (cont.)

- **Lax-Wendroff:**

$$\begin{aligned} g(\xi) &= 1 - \frac{1}{2}\nu\left(e^{i\xi h} - e^{-i\xi h}\right) + \frac{1}{2}\nu^2\left(e^{i\xi h} - 2 + e^{-i\xi h}\right) \\ &= 1 - i\nu \sin(\xi h) + \nu^2\left(\cos(\xi h) - 1\right) \\ &= 1 - i\nu\{2\sin(\xi h/2)\cos(\xi h/2)\} - \nu^2\{2\sin^2(\xi h/2)\}. \end{aligned}$$

$\Longrightarrow |g(\xi)|^2 = 1 - 4\nu^2(1-\nu^2)\sin^4(\xi h/2)$

$\because 0 \le \sin^4(\xi h/2) \le 1$ for all $\xi \Longrightarrow$ stable if $|\nu| \le 1$.

- **Leapfrog:**

$$g(\xi)^2 = 1 - 2\nu i \sin(\xi h) g(\xi),$$

$\Longrightarrow$ stable if $|\nu| < 1$ (cf. Example 7.7).

Characteristic tracing and interpolation

Consider the case $0 < ak/h \leq 1$ and the value $u(x_j, t_{n+1})$. Tracing the characteristic back over time step k from the grid point x_j results in the picture shown in figure (a) below.

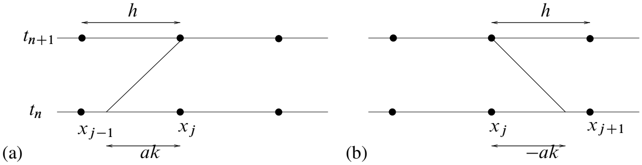
$$u(x_j, t_{n+1}) = u(x_j - ak, t_n),$$

where $x_{j-1} < x_j - ak < x_j$.

- Find U_j^{n+1} by linear interpolation between U_{j-1}^n and U_j^n:

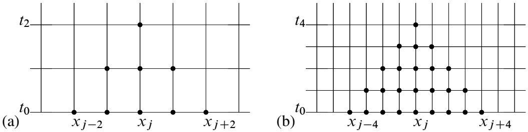
$$\begin{aligned} U_j^{n+1} &:= p(x_j - ak) = U_j^n + ((x_j - ak) - x_j)\left(\frac{U_j^n - U_{j-1}^n}{h}\right) \\ &= U_j^n - \frac{ak}{h}\left(U_j^n - U_{j-1}^n\right) \Longrightarrow \text{first order upwind.} \end{aligned}$$

- Quadratic interpolation U_{j-1}^n, U_j^n, and $U_{j+1}^n \Rightarrow$ Lax-Wendroff
- Quadratic interpolation U_{j-2}^n, U_{j-1}^n, and U_j^n $(0 < ak/h \leq 2) \Rightarrow$ Beam-Warming



Domain of dependence

- As an example, we consider the advection equation $u_t + u_x = 0$. The solution $u(X,T)$ depends on the initial data $\eta(x)$ at only a single point $x = X - aT$. We say the domain of dependence of the point (X,T) is $\mathcal{D}(X,T) = \{X - aT\}$. For the hyperbolic system (Section 10.10), $u_t + Au_x = 0$, then we can show that $\mathcal{D}(X,T) := \{X - \lambda_p T : \ p = 1, \cdots, s\}$, where we assume that λ_p, $1 \le p \le s$, are distinct real eigenvalues of A.
- For a finite difference method, we define the domain of dependence of a grid point (x_j, t_n) to be the set of grid points x_i at the initial time $t = 0$ with the property that the initial value U_i^0 at x_i has an effect on the solution U_j^n. e.g., for Lax-Wendroff, $D(x_j, t_2) = \{x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}\}$ and $D(x_j, t_4) = \{x_{j-4}, x_{j-3}, \cdots, x_j, \cdots, x_{j+3}, x_{j+4}\}$.



Numerical domain of dependence of a grid point when using a 3-point explicit method (e.g., Lax-Wendroff). (a) $t_n = t_2$; (b) $t_n = t_4$.

The Courant-Friedrichs-Lewy (CFL) condition

- If $k/h \equiv r$ fixed, then the numerical domain of dependence of the point (X, T) will fill in the interval $[X - T/r, X + T/r]$. This region must contain the true domain of dependence $\mathcal{D}$ for the PDE. e.g., for the advection equation $u_t + u_x = 0$, we have

$$X - \frac{T}{r} \leq X - aT \leq X + \frac{T}{r} \quad \Longleftrightarrow \quad |a| \leq 1/r \text{ or } \left|\frac{ak}{h}\right| \leq 1.$$

- **Courant-Friendrichs-Lewy (CFL) condition:**
  A numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as $k, h \to 0^+$. (Courant, Friedrichs and Lewy, Math. Ann.,1928).

- For the Lax-Friedrichs, leapfrog, and Lax-Wendroff methods the condition on k and h required by the CFL condition is exactly the stability restriction we derived earlier.

- It is important to note that in general the CFL condition is only *a necessary condition*. If it is violated, then the method cannot be convergent. If it is satisfied, then the method *might be convergent*.

Some remarks

- The forward difference method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(U_{j+1}^n - U_{j-1}^n\right)$$

is always unstable even the CFL condition $|ak/h| \leq 1$ is satisfied.

- The Beam-Warming method ($a > 0$),

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}\left(3U_j^n - 4U_{j-1}^n + U_{j-2}^n\right) + \frac{a^2k^2}{2h^2}\left(U_j^n - 2U_{j-1}^n + U_{j-2}^n\right)$$

has a 3-point one-sided stencil. The CFL condition is satisfied if $0 < ak/h \leq 2$.

Numerical example

- Consider the advection equation $u_t + u_x = 0$ with the periodic boundary condition and the initial data at time $t = 0$ given by

$$u(x,0) = \eta(x) = \exp\{-20(x-2)^2\} + \exp\{-(x-5)^2\}.$$

- Computational domain: $0 \le x \le 25$, $T = 17$, so the exact solution is simply the initial data shifted by 17 units.
- $h = 0.05$, $k = 0.8h$, i.e., the Courant number is $ak/h = 0.8$.

Some numerical results