MA 7007: Numerical Solution of Differential Equations I
Zero-Stability and Convergence for IVPs

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Convergence

Consider the IVP: $u'(t) = f(u(t), t)$ for $t > 0$ and $u(0) = \eta$.

- Convergence of a numerical method for the IVP: Fix time $T > 0$ and then consider the error in our approximation to $u(T)$ using time step $k > 0$. Let $N = T/k$. The convergence means that

$$\lim_{k \to 0,\ Nk = T} U^N = u(T). \qquad (1)$$

  (Note that N increases as $k \to 0$)

- For an r-step method, we need r starting values, $U^0, U^1, \cdots, U^{r-1}$ and these values will typically depend on k. We require

$$\lim_{k \to 0} U^\nu = \eta \quad \text{for } \nu = 0, 1, \cdots r - 1. \qquad (2)$$

- To speak of a method being convergent in general, we require that it converges on all problems in a reasonably large class with all reasonable starting values.

Definition of convergence

An r-step method is said to be convergent if applying the method to any ODE $u'(t) = f(u(t), t)$ for $t > 0$ with $f(u, t)$ Lipschitz continuous in u, and with any set of starting values satisfying (2), we obtain (1) for every fixed $T > 0$ at which the ODE has a unique solution.

In what follows, we consider the simple scalar linear equation of the form

$$u'(t) = \lambda u(t) + g(t), \quad \lambda \in \mathbb{R},$$

with initial value

$$u(t_0) = \eta.$$

By Duhamel's principle, the exact solution is then given by

$$u(t) = e^{\lambda(t-t_0)}\eta + \int_{t_0}^{t} e^{\lambda(t-\tau)} g(\tau) d\tau.$$

Euler's method on linear problems

Applying (forward) Euler's method to the simple scalar linear equation, we obtain

$$U^{n+1} = U^n + k(\lambda U^n + g(t_n)) = (1 + k\lambda)U^n + kg(t_n).$$

The LTE is given by

$$\begin{aligned}\tau^n &= \frac{u(t_{n+1}) - u(t_n)}{k} - \Big(\lambda u(t_n) + g(t_n)\Big) \\ &= \Big(u'(t_n) + \frac{1}{2}ku''(t_n) + O(k^2)\Big) - u'(t_n) \\ &= \frac{1}{2}ku''(t_n) + O(k^2),\end{aligned}$$

which implies that

$$u(t_{n+1}) = (1 + k\lambda)u(t_n) + kg(t_n) + k\tau^n.$$

Define $E^{n+1} := U^{n+1} - u(t_{n+1})$. Then we have the error equation

$$E^{n+1} = (1 + k\lambda)E^n - k\tau^n.$$

Convergence analysis

Applying the recursion repeatedly, we have the discrete form of Duhamel's principle,

$$
\begin{aligned}
E^n &= (1+k\lambda)E^{n-1} - k\tau^{n-1} \\
&= (1+k\lambda)\Big((1+k\lambda)E^{n-2} - k\tau^{n-2}\Big) - k\tau^{n-1} \\
&= \cdots \\
&= (1+k\lambda)^n E^0 - k\sum_{m=1}^{n}(1+k\lambda)^{n-m}\tau^{m-1}.
\end{aligned}
$$

Since

$$|1+k\lambda| \le e^{k|\lambda|} \qquad (\text{using } e^x = 1 + x + \frac{x^2}{2!} + \cdots),$$

we have

$$(1+k\lambda)^{n-m} \le e^{(n-m)k|\lambda|} \le e^{nk|\lambda|} \le e^{|\lambda|T},$$

where we consider the time interval $t_0 := 0 \le t \le T$ and $t_n = nk \le T$.

Convergence analysis (continued)

It then follows that

$$\begin{aligned} |E^n| &\leq e^{|\lambda|T}\left(|E^0| + k\sum_{m=1}^{n}|\tau^{m-1}|\right) \\ &\leq e^{|\lambda|T}\left(|E^0| + nk\max_{1\leq m\leq n}|\tau^{m-1}|\right). \end{aligned}$$

Let $N = T/k$ and set $\|\tau\|_\infty := \max_{0\leq n\leq N-1}|\tau^n|$. Then

$$\|\tau\|_\infty \leq \frac{k}{2}\|u''\|_\infty + O(k^2) = O(k),$$

where $\|u''\|_\infty := \max_{0\leq t\leq T}|u''(t)|$. Then for $nk \leq T$, we have

$$|E^n| \leq e^{|\lambda|T}(|E^0| + T\|\tau\|_\infty) = e^{|\lambda|T}T\|\tau\|_\infty = O(k).$$

Hence, (forward) Euler's method converges and is first order accurate.

Euler's method on nonlinear problems

We consider the IVP: $u'(t) = f(u)$ for $t > 0$ and $u(0) = \eta$. We assume that f is Lipschitz continuous in u on some domain. Then Euler's method for the IVP takes the form

$$U^{n+1} = U^n + kf(U^n)$$

and the LTE is defined by

$$\tau^n = \frac{1}{k}\Big(u(t_{n+1}) - u(t_n)\Big) - f(u(t_n)) = \frac{1}{2}ku''(t_n) + O(k^2).$$

So the true solution satisfies

$$u(t_{n+1}) = u(t_n) + kf(u(t_n)) + k\tau^n$$

and then

$$E^{n+1} = E^n + k\Big(f(U^n) - f(u(t_n))\Big) - k\tau^n.$$

Convergence analysis

Since f is Lipschitz continuous, we get

$$|f(U^n) - f(u(t_n))| \leq L|U^n - u(t_n)| = L|E^n|$$

and then

$$\begin{aligned}
|E^{n+1}| &\leq |E^n| + kL|E^n| + k|\tau^n| \\
&= (1+kL)|E^n| + k|\tau^n| \\
&= (1+kL)\Big((1+kL)|E^{n-1}| + k|\tau^{n-1}|\Big) + k|\tau^n| \\
&= \cdots \\
&= (1+kL)^n|E^0| + k\sum_{m=1}^{n}(1+kL)^{n-m}|\tau^{m-1}|.
\end{aligned}$$

Since $E^0 = 0$, we have

$$|E^n| \leq e^{LT}T\|\tau\|_\infty = O(k)$$

for all n with $nk \leq T$. Hence, (forward) Euler's method converges and is first order accurate.