MA 5037: Optimization Methods and Applications
Convex Optimization

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Convex optimization problem

- **Definition:** (general form) *A convex optimization problem (convex problem) is a problem consisting of minimizing a convex function over a convex set in the form:*

$$\min f(\boldsymbol{x}) \quad \underline{\text{subject to}} \; \boldsymbol{x} \in C,$$

where C is a convex set and f is a convex function over C.

- **Definition:** *(convex optimization problems in functional form)*

$$\begin{array}{rll} \min & f(\boldsymbol{x}) & \\ \text{subject to} & g_i(\boldsymbol{x}) \le 0, & i = 1, 2, \cdots, m, \\ & h_j(\boldsymbol{x}) = 0, & j = 1, 2, \cdots, p, \end{array}$$

where $f, g_1, \cdots, g_m : \mathbb{R}^n \to \mathbb{R}$ *are convex functions and* $h_1, \cdots, h_p : \mathbb{R}^n \to \mathbb{R}$ *are affine functions.*

- **Note:** *The above problem does fit into the general form. In fact,*

$$C = \left(\cap_{i=1}^{m} \text{Lev}(g_i, 0) \right) \cap \left(\cap_{j=1}^{p} \{ \boldsymbol{x} : h_j(\boldsymbol{x}) = 0 \} \right)$$

is convex and closed, since g_i, h_j *are continuous on* $\mathbb{R}^n$ *and the inverse images of closed sets under continuous functions are closed sets.*

“local = global” in convex optimization problem

- **Theorem** (local = global): *Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function defined over the convex set C. Let $\boldsymbol{x}^* \in C$ be a local minimum of f over C. Then $\boldsymbol{x}^*$ is a global minimum of f over C.*

  Proof: Since $\boldsymbol{x}^* \in C$ is a local minimum of f over C, $\exists\, r > 0$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)$ for any $\boldsymbol{x} \in C \cap B[\boldsymbol{x}^*, r]$. Let $\boldsymbol{x}^* \neq \boldsymbol{y} \in C$ and $\lambda \in (0, 1]$ such that $\boldsymbol{x}^* + \lambda(\boldsymbol{y} - \boldsymbol{x}^*) \in B[\boldsymbol{x}^*, r]$. Since $\boldsymbol{x}^* + \lambda(\boldsymbol{y} - \boldsymbol{x}^*) \in B[\boldsymbol{x}^*, r] \cap C$, it follows that

  $$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}^* + \lambda(\boldsymbol{y} - \boldsymbol{x}^*)) \leq (1 - \lambda) f(\boldsymbol{x}^*) + \lambda f(\boldsymbol{y}).$$

  Thus $\lambda f(\boldsymbol{x}^*) \leq \lambda f(\boldsymbol{y})$, and then $f(\boldsymbol{x}^*) \leq f(\boldsymbol{y})$. $\square$

- **Theorem:** *Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function defined over the convex set C. Let $\boldsymbol{x}^* \in C$ be a local minimum of f over C. Then $\boldsymbol{x}^*$ is a strict global minimum of f over C.*

- **Definition:** *The optimal set of the convex optimization problem is the set of all minimizers, that is,* $X^* := \operatorname{argmin}\{f(\boldsymbol{x}) : \boldsymbol{x} \in C\}$.

Convexity of the optimal set in convex optimization

Theorem: *Let $f : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a convex function defined over the convex set C. Then the set X^* of optimal solutions of the problem $\min\{f(\boldsymbol{x}) : \boldsymbol{x} \in C\}$ is convex. If, in addition, f is strictly convex over C, then there exists at most one optimal solution.*

Proof: If $X^* = \varnothing$, then the result follows trivially.

- Assume that $X^* \neq \varnothing$ and denote the optimal value by f^*. Let $\boldsymbol{x}, \boldsymbol{y} \in X^*$ and $\lambda \in [0, 1]$. Then
$$f(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \leq \lambda f^* + (1 - \lambda) f^* = f^*.$$
Hence, $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in C$ is also an optimal solution and $\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in X^*$, establishing the convexity of X^*.
- Assume that f is strictly convex over C and $X^* \neq \varnothing$. To show that X^* is a singleton, suppose in contradiction that there exist $\boldsymbol{x}, \boldsymbol{y} \in X^*$ such that $\boldsymbol{x} \neq \boldsymbol{y}$. Then $\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y} \in X^* \subseteq C$ and
$$f(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}) < \frac{1}{2} f(\boldsymbol{x}) + \frac{1}{2} f(\boldsymbol{y}) = \frac{1}{2} f^* + \frac{1}{2} f^* = f^*,$$
which is a contradiction to the fact that f^* is the optimal value. □

Maximizing concave functions over convex sets

- **Note:** *Convex optimization problems consist of minimizing convex functions over convex sets, but we will also refer to problems consisting of maximizing concave functions over convex sets as convex problems.*
- **Example:** *The following problem is a convex problem:*

  $$\min(-2x + y) \quad \textit{subject to} \;\; x^2 + y^2 \le 3.$$

  The objective function is linear and thus convex, and the single inequality constraint is the level set $Lev(f, 0)$ of the convex function $f(x, y) = x^2 + y^2 - 3$ and hence convex.
- **Example:** *The following problem is nonconvex:*

  $$\min(x^2 - y) \quad \textit{subject to} \;\; x^2 + y^2 = 3.$$

  The objective function is convex, but the constraint is a nonlinear equality constraint and therefore nonconvex. *Note that the feasible set is the boundary of the ball with center $(0, 0)$ and radius $\sqrt{3}$.*

Linear programming

- *A linear programming (LP) problem* is an optimization problem consisting of minimizing a linear objective function subject to linear equalities and inequalities:

$$\min \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{B}\boldsymbol{x} = \boldsymbol{g},$$

  where $\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{B} \in \mathbb{R}^{p\times n}$, and $\boldsymbol{g} \in \mathbb{R}^p$.

  We remark here that for a vector $\boldsymbol{z} \ge \boldsymbol{0}$ we mean $z_i \ge 0$ for all i.

- *Linear functions are both convex and concave.* Consider the LP problem (called the "standard formulation" in the literature):

$$\max \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0},$$

  a problem of maximizing a convex function over a convex set. *From Theorem 7.42, if the feasible set is nonempty and compact, then $\exists$ at least one optimal solution which is an extreme point of the feasible set ($\Rightarrow$ a basic feasible solution, if $\boldsymbol{A}$ has linearly independent rows).*

- A more general result dropping the *"compactness assumption"* is called the *fundamental theorem of linear programming.*

Convex quadratic problems

Convex quadratic problems are problems consisting of minimizing a convex quadratic function subject to affine constraints. A general form of problems of this class can be written as

$$\min \quad \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} \quad \text{s.t.} \;\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{c},$$

where $\mathbf{0} \preceq \boldsymbol{Q} \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, and $\boldsymbol{c} \in \mathbb{R}^m$.

- *Classification via linear separators:*

  Suppose that we are given two types of points in $\mathbb{R}^n$: (type A) $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m$, (type B) $\boldsymbol{x}_{m+1}, \boldsymbol{x}_{m+2}, \cdots, \boldsymbol{x}_{m+p}$. The objective is to find a linear separator, which is a hyperplane, $H(\boldsymbol{w}, \beta) := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{w}^\top \boldsymbol{x} + \beta = 0\}$, for which the type A and type B points are in its opposite sides:

  $$\boldsymbol{w}^\top \boldsymbol{x}_i + \beta < 0,\ 1 \leq i \leq m, \quad \boldsymbol{w}^\top \boldsymbol{x}_i + \beta > 0,\ m+1 \leq i \leq m+p.$$

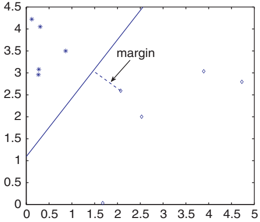
  Our underlying assumption is that the two sets of points are linearly separable, i.e., the set of inequalities has a solution.

- *The problem is not well-defined in the sense that there are many linear separators, and what we seek is in fact a separator that is in a sense farthest as possible from all the points.*

Classification via linear separators

- Margin of the separator: the distance of the separator from the closest point, as illustrated in above figure. Therefore, we have

$$\text{margin} := \min_{i=1,2,\cdots,m+p} \frac{|\boldsymbol{w}^\top \boldsymbol{x}_i + \beta|}{\|\boldsymbol{w}\|}.$$



- *The separation problem will thus consist of finding the separator with the largest margin:*

$$\max\left\{\min_{i=1,2,\cdots,m+p} \frac{|\boldsymbol{w}^\top \boldsymbol{x}_i + \beta|}{\|\boldsymbol{w}\|}\right\}$$
$$\text{s.t. } \boldsymbol{w}^\top \boldsymbol{x}_i + \beta < 0,\ 1 \le i \le m,$$
$$\boldsymbol{w}^\top \boldsymbol{x}_i + \beta > 0,\ m+1 \le i \le m+p.$$

Classification via linear separators (cont'd)

- This is a bad formulation of the problem since it is not convex.
- Note that the problem has a degree of freedom in the sense that if $(\boldsymbol{w}, \beta)$ is an optimal solution, then so is $(\alpha\boldsymbol{w}, \alpha\beta)$ for $\alpha \neq 0$. The problem can then be rewritten as

$$\begin{aligned} &\max \frac{1}{\|\boldsymbol{w}\|} \\ &\text{s.t.} \min_{i=1,2,\cdots,m+p} |\boldsymbol{w}^\top \boldsymbol{x}_i + \beta| = 1, \\ &\quad \boldsymbol{w}^\top \boldsymbol{x}_i + \beta < 0, 1 \leq i \leq m, \boldsymbol{w}^\top \boldsymbol{x}_i + \beta > 0, m+1 \leq i \leq m+p. \end{aligned}$$

The combination of the first equality and the other inequality constraints implies that a valid reformulation is

$$\begin{aligned} &\min \frac{1}{2}\|\boldsymbol{w}\|^2 \\ &\text{s.t.} \min_{i=1,2,\cdots,m+p} |\boldsymbol{w}^\top \boldsymbol{x}_i + \beta| = 1, \\ &\quad \boldsymbol{w}^\top \boldsymbol{x}_i + \beta \leq -1,\ 1 \leq i \leq m, \boldsymbol{w}^\top \boldsymbol{x}_i + \beta \geq 1,\ m+1 \leq i \leq m+p. \end{aligned}$$

Classification via linear separators (cont'd)

- In the above, we also used the fact that maximizing $1/\|\boldsymbol{w}\|$ is the same as minimizing $\frac{1}{2}\|\boldsymbol{w}\|^2$ in the sense that the optimal set stays the same.
- Finally, we remove the problematic "min" equality constraint and obtain the following convex quadratic reformulation:

$$\begin{aligned} &\min \frac{1}{2}\|\boldsymbol{w}\|^2 \\ &\text{s.t. } \boldsymbol{w}^\top \boldsymbol{x}_i + \beta \leq -1,\ 1 \leq i \leq m, \\ &\qquad \boldsymbol{w}^\top \boldsymbol{x}_i + \beta \geq 1,\ m+1 \leq i \leq m+p. \end{aligned}$$

The removal of the "min" constraint is valid since any feasible solution of problem satisfies $\min_{i=1,2,\cdots,m+p} |\boldsymbol{w}^\top \boldsymbol{x}_i + \beta| \geq 1$.

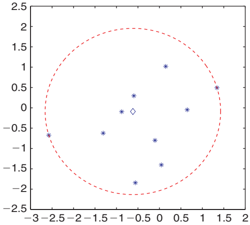
If $(\boldsymbol{w}, \beta)$ is an optimal solution, then equality must be satisfied. Otherwise, if $\min\limits_{i=1,2,\cdots,m+p} |\boldsymbol{w}^\top \boldsymbol{x}_i + \beta| > 1$, then a better solution (with lower objective function value) will be $\frac{1}{\alpha}(\boldsymbol{w}, \beta)$, where $\alpha = \min\limits_{i=1,2,\cdots,m+p} |\boldsymbol{w}^\top \boldsymbol{x}_i + \beta|$.

Chebyshev center of a set of points

- Suppose that we are given m points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$ in $\mathbb{R}^n$. The objective is to find the center of the minimum radius closed ball containing all the points. *This ball is called the Chebyshev ball and the corresponding center is the Chebyshev center.*
- Let r be the radius and $\boldsymbol{x}$ be the center. The problem can be written as "$\min_{\boldsymbol{x},r}\ r$ s.t. $\boldsymbol{a}_i \in B[\boldsymbol{x}, r],\ i = 1, 2, \cdots, m$". That is,

$$\min_{\boldsymbol{x},r}\ r \quad \text{s.t. } \|\boldsymbol{a}_i - \boldsymbol{x}\| \le r,\ i = 1, 2, \cdots, m.$$

This is a convex problem since it consists of minimizing a linear function subject to convex inequality constraints.



Hidden convexity in trust region subproblems

- There are several situations in which a certain problem is not convex but nonetheless can be recast as a convex optimization problem, called *"hidden convexity."*
- The trust region subproblem consists of minimizing a quadratic function (not necessarily convex) subject to an Euclidean norm constraint:

$$(\mathrm{TRS}) \quad \min\{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c : \|\boldsymbol{x}\|^2 \le 1\},$$

  where $\boldsymbol{b} \in \mathbb{R}^n$, $c \in \mathbb{R}$, and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ symmetric matrix which is not necessarily positive semidefinite.
- *Since the objective function is (possibly) nonconvex, problem (TRS) is (possibly) nonconvex.* This is an important class of problems arising, for example, as a subroutine in *trust region methods,* hence the name of this class of problems.

Transforming TRS into a convex optimization problem

- By the spectral decomposition theorem, $\exists$ an orthogonal matrix $\boldsymbol{U}$ and a diagonal matrix $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \cdots, d_n)$ such that $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\top$, and TRS can be rewritten as

$$(\text{TRS}) \quad \min\{\boldsymbol{x}^\top \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\top \boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{U}\boldsymbol{U}^\top \boldsymbol{x} + c : \|\boldsymbol{U}^\top \boldsymbol{x}\|^2 \leq 1\},$$

where $\|\boldsymbol{U}^\top \boldsymbol{x}\| = \|\boldsymbol{x}\|$. Let $\boldsymbol{y} = \boldsymbol{U}^\top \boldsymbol{x}$, we have

$$\min\{\boldsymbol{y}^\top \boldsymbol{D}\boldsymbol{y} + 2\boldsymbol{b}^\top \boldsymbol{U}\boldsymbol{y} + c : \|\boldsymbol{y}\|^2 \leq 1\}.$$

- Denoting $\boldsymbol{v} = (v_1, v_2, \cdots, v_n) := \boldsymbol{U}^\top \boldsymbol{b}$, we obtain

$$\min\left(\sum_{i=1}^n d_i y_i^2 + 2\sum_{i=1}^n v_i y_i + c\right) \quad \text{s.t.} \ \sum_{i=1}^n y_i^2 \leq 1.$$

The problem is still nonconvex, since some of the d_i's might be < 0.

A Lemma

Lemma: *Let $\boldsymbol{y}^*$ be an optimal solution of the above problem, then $v_i y_i^* \le 0$, for all $i = 1, 2, \cdots, n$.*

Proof: Denote the objective function by

$$f(\boldsymbol{y}) := \sum_{j=1}^{n} d_j y_j^2 + 2 \sum_{j=1}^{n} v_j y_j + c.$$

Let $i \in \{1, 2, \cdots, n\}$. Define the vector

$$\tilde{y}_j = \begin{cases} y_j^*, & j \neq i, \\ -y_j^*, & j = i. \end{cases}$$

Then $\tilde{\boldsymbol{y}}$ is a feasible solution. Since $\boldsymbol{y}^*$ is an optimal solution, $f(\boldsymbol{y}^*) \le f(\tilde{\boldsymbol{y}})$, which is the same as

$$\sum_{j=1}^{n} d_j (y_j^*)^2 + 2 \sum_{j=1}^{n} v_j y_j^* + c \le \sum_{j=1}^{n} d_j (\tilde{y}_j)^2 + 2 \sum_{j=1}^{n} v_j \tilde{y}_j + c.$$

Thus, we have $2 v_i y_i^* \le 2 v_i (-y_i^*)$, which implies that $v_i y_i^* \le 0$. $\square$

Transforming TRS into a convex optimization problem

- By the Lemma, for any optimal solution $\boldsymbol{y}^*$, $\operatorname{sgn}(y_i^*) = -\operatorname{sgn}(v_i)$ when $v_i \neq 0$, where the sgn function is defined to be

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

  When $v_i = 0$, we have the property that both $\boldsymbol{y}^*$ and $\tilde{\boldsymbol{y}}$ are optimal. Hence the sign of y_i^* can be chosen arbitrarily. As a consequence, we can make the change of variables $y_i = -\operatorname{sgn}(v_i)\sqrt{z_i}$, $z_i \geq 0$, and the problem (page 13) becomes

$$\min\left(\sum_{i=1}^{n} d_i z_i - 2\sum_{i=1}^{n} |v_i|\sqrt{z_i} + c\right) \quad \text{s.t. } \sum_{i=1}^{n} z_i \leq 1, \quad z_1, z_2, \cdots, z_n \geq 0.$$

- *This is a convex optimization problem since the constraints are linear and the objective function is a sum of linear terms and positive multipliers of the convex functions* $-\sqrt{z_i}$.

Orthogonal projection operator

- **Definition:** *Given a nonempty closed convex set C, the orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by*

$$P_C(\boldsymbol{x}) := \operatorname{argmin}\{\|\boldsymbol{y}-\boldsymbol{x}\|^2 : \boldsymbol{y} \in C\}. \qquad (\star)$$

  In other words, the orthogonal projection operator with input $\boldsymbol{x}$ returns the vector in C that is closest to $\boldsymbol{x}$. Thus, we have the distance function $d(\boldsymbol{x}, C) := \min_{\boldsymbol{y}\in C} \|\boldsymbol{x}-\boldsymbol{y}\| = \|\boldsymbol{x} - P_C(\boldsymbol{x})\|$.

- The orthogonal projection operator is defined as a solution of a convex optimization problem, specifically, a minimization of a convex quadratic function subject to a convex feasibility set.

- **Theorem:** *(first projection theorem) Let C be a nonempty closed convex set. Then problem $(\star)$ has a unique optimal solution.*

  Proof: Since the objective function in $(\star)$ is a quadratic function with a positive definite matrix, it follows that it is coercive and hence that the problem has at least one optimal solution. Since the objective function is strictly convex, it follows that there exists only one optimal solution. □

Projection on the nonnegative orthant

- Computing the orthogonal projection operator might be a difficult task. Below are some examples of simple sets.
- Let $C = \mathbb{R}^n_+$. To compute the orthogonal projection of $\boldsymbol{x} \in \mathbb{R}^n$ onto C, we need to solve the convex optimization problem:

$$\min \sum_{i=1}^{n} (y_i - x_i)^2, \quad \text{s.t. } y_1, y_2, \cdots, y_n \geq 0.$$

Since this problem is separable, meaning that the objective function is a sum of functions of each of the variables, and the constraints are separable in the sense that each of the variables has its own constraint, it follows that the ith component of the optimal solution $\boldsymbol{y}^*$ is the optimal solution of the problem

$$\min\{(y_i - x_i)^2 : y_i \geq 0\}.$$

Thus, the solution is given by $y_i^* = [x_i]_+ := x_i$ *if* $x_i \geq 0$ *and* 0 *if* $x_i < 0$. *Therefore,* $P_{\mathbb{R}^n_+}(\boldsymbol{x}) = [\boldsymbol{x}]_+ := ([x_1]_+, [x_2]_+, \cdots, [x_n]_+)^\top$.

Projection on boxes

- A box is a subset of $\mathbb{R}^n$ of the form

$$B = [\ell_1, u_1] \times [\ell_2, u_2] \times \cdots \times [\ell_n, u_n] = \{\boldsymbol{x} \in \mathbb{R}^n : \ell_i \le x_i \le u_i\}.$$

  We will also allow some of the u_i's to be equal to ∞ and some of the ℓ_i's to be equal to $-\infty$, and in these cases we will assume that ∞ or $-\infty$ are not actually contained in the intervals.

- A similar separability argument as the one used in the previous example, one can show that the orthogonal projection is given by $\boldsymbol{y} := (y_1, y_2, \cdots, y_n)^\top = P_B(\boldsymbol{x})$, where for $i = 1, 2, \cdots, n$,

$$y_i = \begin{cases} u_i, & x_i \ge u_i, \\ x_i, & \ell_i < x_i < u_i, \\ \ell_i, & x_i \le \ell_i. \end{cases}$$

Projection onto balls

- Let $C = B[\mathbf{0}, r] = \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y}\| \le r\}$. The minimization problem is given by

$$\min\{\underbrace{\|\boldsymbol{y}-\boldsymbol{x}\|^2}_{:=f(\boldsymbol{y})} : \|\boldsymbol{y}\|^2 \le r^2\}.$$

- If $\|\boldsymbol{x}\| \le r$, then $\boldsymbol{y} = \boldsymbol{x}$ is the optimal solution. When $\|\boldsymbol{x}\| > r$, the optimal solution must belong to the boundary of the ball since otherwise, $\nabla f(\boldsymbol{y}) = \mathbf{0} \Rightarrow 2(\boldsymbol{y}-\boldsymbol{x}) = \mathbf{0} \Rightarrow \boldsymbol{y} = \boldsymbol{x} \notin C$. $(\rightarrow\leftarrow)$
- Therefore, the minimization problem is equivalent to

$$\min\{\|\boldsymbol{y}-\boldsymbol{x}\|^2 : \|\boldsymbol{y}\|^2 = r^2\},$$

or equivalently

$$\min\{-2\boldsymbol{x}^\top\boldsymbol{y} + r^2 + \|\boldsymbol{x}\|^2 : \|\boldsymbol{y}\|^2 = r^2\}, \text{ or } \min\{-2\boldsymbol{x}^\top\boldsymbol{y} : \|\boldsymbol{y}\|^2 = r^2\}.$$

By the CS inequality, $-2\boldsymbol{x}^\top\boldsymbol{y} \ge -2\|\boldsymbol{x}\|\|\boldsymbol{y}\| = -2r\|\boldsymbol{x}\|$, and this lower bound is attained at $\boldsymbol{y} = r\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. Thus, $P_{B[\mathbf{0},r]}(\boldsymbol{x}) = \boldsymbol{x}$ if $\|\boldsymbol{x}\| \le r$ and $P_{B[\mathbf{0},r]}(\boldsymbol{x}) = r\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ if $\|\boldsymbol{x}\| > r$.

CVX: a MATLAB-based modeling system

- CVX is a MATLAB-based modeling system for convex optimization problems. It was created by Michael Grant and Stephen Boyd.
- A comprehensive and complete guide can be found at the CVX website `http://CVXr.com`
- The basic structure of a CVX program is as follows:

```
cvx_begin
{variables declaration}
minimize({objective function})
      or maximize({objective function})
subject to
{constraints}
cvx_end
```