MA 5037: Optimization Methods and Applications
Unconstrained Optimization

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First version: May 17, 2018/Last updated: June 13, 2025

Global minimum and global maximum

Definition: *Let $f : S \to \mathbb{R}$ be a real-valued function defined on a nonempty set $S \subseteq \mathbb{R}^n$.*

(1) $\boldsymbol{x}^* \in S$ *is called a global minimum point (minimizer) of f over S if* $f(\boldsymbol{x}^*) \le f(\boldsymbol{x}), \forall\, \boldsymbol{x} \in S$.

(2) $\boldsymbol{x}^* \in S$ *is called a strict global minimum point (minimizer) of f over S if* $f(\boldsymbol{x}^*) < f(\boldsymbol{x}), \forall\, \boldsymbol{x} \in S$ *and* $\boldsymbol{x} \neq \boldsymbol{x}^*$. *(for short,* $\forall\, \boldsymbol{x}^* \neq \boldsymbol{x} \in S$*)*

(3) $\boldsymbol{x}^* \in S$ *is called a global maximum point (maximizer) of f over S if* $f(\boldsymbol{x}) \le f(\boldsymbol{x}^*), \forall\, \boldsymbol{x} \in S$.

(4) $\boldsymbol{x}^* \in S$ *is called a strict global maximum point (maximizer) of f over S if* $f(\boldsymbol{x}) < f(\boldsymbol{x}^*), \forall\, \boldsymbol{x}^* \neq \boldsymbol{x} \in S$.

(5) *The set S on which the optimization of f is performed is called the feasible set, and any point $\boldsymbol{x} \in S$ is called a feasible solution.*

Note: We will frequently omit the adjective *"global"*.

Minimal value and maximal value of f over S

Definition: *Let $f : S \to \mathbb{R}$ be a real-valued function defined on a nonempty set $S \subseteq \mathbb{R}^n$.*

(1) *$\boldsymbol{x}^* \in S$ is called a global optimum of f over S if it is either a global minimizer or a global maximizer.*

(2) *The minimal value of f over $S := \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\}$. If $\boldsymbol{x}^* \in S$ is a global minimum of f over S, then $\inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\} = f(\boldsymbol{x}^*)$.*

(3) *The maximal value of f over $S := \sup\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\}$. If $\boldsymbol{x}^* \in S$ is a global maximum of f over S, then $\sup\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\} = f(\boldsymbol{x}^*)$.*

(4) *The set of all global minimizers of f over S is denoted by*

$$\operatorname{argmin}\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\}.$$

The set of all global maximizers of f over S is denoted by

$$\operatorname{argmax}\{f(\boldsymbol{x}) : \boldsymbol{x} \in S\}.$$

Example 1

Find the global minimum and maximum points of $f(x,y) = x + y$ *over* $S = B[\mathbf{0}, 1] = \{(x,y)^\top : x^2 + y^2 \leq 1\}$.

Solution:

- By the Cauchy-Schwarz inequality, for any $(x,y)^\top \in S$, we have

$$x + y = (x,y)\begin{bmatrix}1\\1\end{bmatrix} \leq \sqrt{x^2+y^2}\sqrt{1^2+1^2} \leq \sqrt{2}.$$

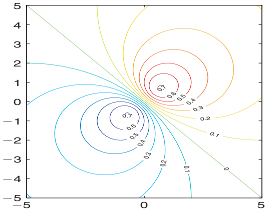
  Therefore, the maximal value of f over S is upper bounded by $\sqrt{2}$. Note that $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in S$ and $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \sqrt{2}$ and this is the *only* point that attains this value. Thus, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is the *strict* global maximum point of f over S, and the maximal value is $\sqrt{2}$.
- Similarly, we can show that $-(x+y) \leq \sqrt{2} \Longrightarrow x + y \geq -\sqrt{2}$. Thus, $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ is the strict global minimum point of f over S, and the minimal value is $-\sqrt{2}$.

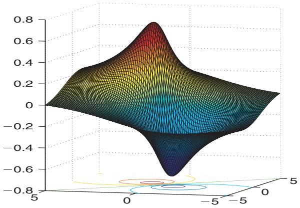
Example 2

Consider the following 2-D function defined over the entire space:

$$f(x,y) = \frac{x+y}{x^2+y^2+1}.$$

The contour and surface plots of the function are given below:





- The global maximizer $= (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, the maximal value $= \frac{1}{\sqrt{2}}$.
- The global minimizer $= (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$, the minimal value $= \frac{-1}{\sqrt{2}}$.

The proof of these facts will be given later.

Local minimum and local maximum

Definition: *Let $f : S \to \mathbb{R}$ be a real-valued function defined on a nonempty set $S \subseteq \mathbb{R}^n$.*

(1) $\boldsymbol{x}^* \in S$ *is called a local minimum point of f over S if* $\exists\, r > 0$ *such that* $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}), \forall\, \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*, r)$.

(2) $\boldsymbol{x}^* \in S$ *is called a strict local minimum point of f over S if* $\exists\, r > 0$ *such that* $f(\boldsymbol{x}^*) < f(\boldsymbol{x}), \forall\, \boldsymbol{x}^* \neq \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*, r)$.

(3) $\boldsymbol{x}^* \in S$ *is called a local maximum point of f over S if* $\exists\, r > 0$ *such that* $f(\boldsymbol{x}) \leq f(\boldsymbol{x}^*), \forall\, \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*, r)$.

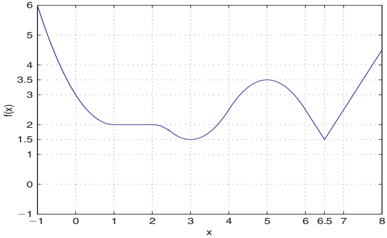
(4) $\boldsymbol{x}^* \in S$ *is called a strict local maximum point of f over S if* $\exists\, r > 0$ *such that* $f(\boldsymbol{x}) < f(\boldsymbol{x}^*), \forall\, \boldsymbol{x}^* \neq \boldsymbol{x} \in S \cap B(\boldsymbol{x}^*, r)$.

Example

Consider the following 1-D function defined over $[-1, 8]$:

$$f(x) = \begin{cases} (x-1)^2 + 2, & -1 \le x \le 1, \\ 2, & 1 \le x \le 2, \\ -(x-2)^2 + 2, & 2 \le x \le 2.5, \\ (x-3)^2 + 1.5, & 2.5 \le x \le 4, \\ -(x-5)^2 + 3.5, & 4 \le x \le 6, \\ -2x + 14.5, & 6 \le x \le 6.5, \\ 2x - 11.5, & 6.5 \le x \le 8. \end{cases}$$

Classify each of the points $x = -1, 1, 2, 3, 5, 6.5, 8$ as strict/nonstrict, global/local, minimum/maximum points.



First order optimality condition for local optimum points

Theorem: *Let $f : U \to \mathbb{R}$ be a function defined on a set $\varnothing \neq U \subseteq \mathbb{R}^n$. Assume that $\boldsymbol{x}^* \in int(U)$ is a local optimum point and that all the partial derivatives of f exist at $\boldsymbol{x}^*$. Then $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$.* (Fermat's theorem in 1D)

Proof: Given $1 \leq i \leq n$, we define the function $g_i(t) := f(\boldsymbol{x}^* + t\boldsymbol{e}_i)$. Then g_i is differentiable at $t = 0$ and $g_i'(0) = \nabla f(\boldsymbol{x}^*) \cdot \boldsymbol{e}_i = \frac{\partial f}{\partial x_i}(\boldsymbol{x}^*)$. Since $\boldsymbol{x}^*$ is a local optimum point of f, it follows that $t = 0$ is a local optimum of g_i. By Fermat's theorem, we have $g_i'(0) = 0$, which implies that $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$. □

Note: First order optimality condition is only *a necessary condition.*

Definition: *Let $f : U \to \mathbb{R}$ be a function defined on a set $\varnothing \neq U \subseteq \mathbb{R}^n$. Assume that $\boldsymbol{x}^* \in int(U)$ and all the partial derivatives of f exist over some neighborhood of $\boldsymbol{x}^*$. If $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$, then $\boldsymbol{x}^*$ is called a stationary point of f.*

Positive definiteness (review)

- Let $A \in \mathbb{C}^{n\times n}$ be a square matrix and $x, y \in \mathbb{C}^n$. Define $x^* := \overline{x}^\top$, $(x, y) := y^* x \in \mathbb{C}$. *Then* $(Ax, x) = x^* A x$ *is called a quadratic form.*
- **Definition:** Let $A \in \mathbb{C}^{n\times n}$.

$$A \text{ is positive definite} \iff (Ax, x) > 0, \quad \forall\, 0 \neq x \in \mathbb{C}^n.$$

- **Note 1:** $A = A^* (:= \overline{A}^\top) \iff (Ax, x) \in \mathbb{R}, \forall\, x \in \mathbb{C}^n$.
- **Note 2:** If $A \in \mathbb{C}^{n\times n}$ is positive definite, then $A = A^*$. (by Note 1)
- **Note 3:** Let $A \in \mathbb{R}^{n\times n}$. A is positive definite $\iff A = A^\top$ and $(Ax, x) > 0, \forall\, 0 \neq x \in \mathbb{R}^n$.

  Proof: $(\Rightarrow)$ Trivial!

  $(\Leftarrow)$ Let $0 \neq x := x_1 + ix_2 \in \mathbb{C}^n$. Then $x_1 \neq 0$ or $x_2 \neq 0$.

  $\therefore (A(x_1 + ix_2), (x_1 + ix_2)) = (Ax_1, x_1) - i(Ax_1, x_2) + i(Ax_2, x_1) + (Ax_2, x_2)$

  $\because -i(Ax_1, x_2) = -i(x_1, A^* x_2) = -i(x_1, A^\top x_2) = -i(x_1, Ax_2) = -i(Ax_2, x_1)$

  $\therefore (A(x_1 + ix_2), (x_1 + ix_2)) = (Ax_1, x_1) + (Ax_2, x_2) > 0$
- **Note 4:** Let $A \in \mathbb{C}^{n\times n}$ and $A = A^*$. Then A is positive definite $\iff$ all of its eigenvalues are real and positive.

Proof of Note 1

$(\Rightarrow)\ \because\ (Ax,x) = x^*Ax = (Ax)^*x = (x,Ax) = \overline{(Ax,x)}, \forall\, x \in \mathbb{C}^n$
$\therefore\ (Ax,x) \in \mathbb{R}, \forall\, x \in \mathbb{C}^n$

$(\Leftarrow)\ \forall\, x,y \in \mathbb{C}^n$, we have
$\mathbb{R} \ni (x+y)^*A(x+y) = x^*Ax + y^*Ay + x^*Ay + y^*Ax.$
$\therefore\ x^*Ay + y^*Ax \in \mathbb{R}$

- Let $x = e_j \in \mathbb{R}^n, y = e_k \in \mathbb{R}^n$. Then $\mathbb{R} \ni x^*Ay + y^*Ax = a_{jk} + a_{kj}$
  $\therefore\ Im(a_{jk}) = -Im(a_{kj})$
  $\therefore\ a_{jk} := a + bi$ and $a_{kj} := c - bi$ for some $a, b, c \in \mathbb{R}$
- Let $x = ie_j \in \mathbb{C}^n, y = e_k \in \mathbb{R}^n$. Then

  $$\mathbb{R} \ni x^*Ay + y^*Ax = -ia_{jk} + ia_{kj} = (-ia + b) + (ci + b) = (c-a)i + 2b.$$

  $\therefore\ c = a$. Then $a_{jk} := a + bi = \overline{a - bi} = \overline{a_{kj}}$
  $\therefore\ A = \overline{A}^\top = A^*$

Positive definiteness

- **Definition:** *A symmetric matrix* $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ *is called positive semidefinite, denoted by* $\boldsymbol{A} \succeq \mathbf{0}$*, if* $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \geq 0, \forall\, \boldsymbol{x} \in \mathbb{R}^n$.
- **Definition:** *A symmetric matrix* $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ *is called positive definite, denoted by* $\boldsymbol{A} \succ \mathbf{0}$*, if* $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0, \forall\, \mathbf{0} \neq \boldsymbol{x} \in \mathbb{R}^n$.
- **Example:** Let $A := \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. $\forall\, \boldsymbol{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, we have

$$\begin{aligned} \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} &= [x_1, x_2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 - 2x_1x_2 + x_2^2 = x_1^2 + (x_1 - x_2)^2 \geq 0. \end{aligned}$$

  Since $x_1^2 + (x_1 - x_2)^2 = 0$ iff $x_1 = x_2 = 0$, we have $\boldsymbol{A} \succ \mathbf{0}$.

- **Example:** Let $A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. One can show that A is not positive definite. *Hint: consider* $\boldsymbol{x} = (1, -1)^\top$

The diagonal components of a positive definite matrix

- **Note:** *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A} \succ \mathbf{0}$. Then the diagonal elements of $\boldsymbol{A}$ are positive.* *Proof:* $A_{ii} = \boldsymbol{e}_i^\top \boldsymbol{A}\boldsymbol{e}_i > 0, \; \forall\, i.$ $\square$
  Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A} \succeq \mathbf{0}$. Then the diagonal elements of $\boldsymbol{A}$ are nonnegative.
- **Definition:** $\boldsymbol{A} \preceq \mathbf{0}$ *(negative semidefinite) iff* $-\boldsymbol{A} \succeq \mathbf{0}$.
  $\boldsymbol{A} \prec \mathbf{0}$ *(negative definite) iff* $-\boldsymbol{A} \succ \mathbf{0}$.
- **Note:** *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A} \prec \mathbf{0}$. Then the diagonal elements of $\boldsymbol{A}$ are negative.*
  Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{A} \preceq \mathbf{0}$. Then the diagonal elements of $\boldsymbol{A}$ are nonpositive.
- **Definition:** *A symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is called indefinite if* $\exists\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *such that* $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0$ *and* $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y} < 0$.
- **Note:** *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix. If there exist positive and negative elements in the diagonal of $\boldsymbol{A}$, then $\boldsymbol{A}$ is indefinite.*
  Proof: Let i and j be the indices such that $A_{ii} > 0$ and $A_{jj} < 0$. Then $\boldsymbol{e}_i^\top \boldsymbol{A}\boldsymbol{e}_i = A_{ii} > 0$ and $\boldsymbol{e}_j^\top \boldsymbol{A}\boldsymbol{e}_j = A_{jj} < 0$. $\square$

Eigenvalue characterization theorem

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then

- *$\boldsymbol{A} \succ \mathbf{0}$ if and only if all its eigenvalues are positive.*

  Proof: By the spectral decomposition theorem, there exist an orthogonal matrix $\boldsymbol{U}$ and a diagonal matrix $\boldsymbol{D} = diag(d_1, \cdots, d_n)$ whose diagonal elements are the eigenvalues of $\boldsymbol{A}$, for which $\boldsymbol{U}^\top \boldsymbol{A}\boldsymbol{U} = \boldsymbol{D}$. For any $\mathbf{0} \neq \boldsymbol{x} \in \mathbb{R}^n$, let $\boldsymbol{y} = \boldsymbol{U}^{-1}\boldsymbol{x}$. Then

  $$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{U}^\top \boldsymbol{A}\boldsymbol{U}\boldsymbol{y} = \boldsymbol{y}^\top \boldsymbol{D}\boldsymbol{y} = \sum_{i=1}^{n} d_i y_i^2.$$

  Therefore, $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0$ for any $\boldsymbol{x} \neq \mathbf{0}$ if and only if $\sum_{i=1}^n d_i y_i^2 > 0$ for any $\boldsymbol{y} \neq \mathbf{0}$.
  (1) For any given i, let $\boldsymbol{y} = \boldsymbol{e}_i$, we have $d_i > 0$, i.e., all eigenvalues are positive.
  (2) If $d_i > 0\ \forall\ i$, then $\sum_{i=1}^n d_i y_i^2 > 0$ for any $\boldsymbol{y} \neq \mathbf{0}$, i.e., $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0$ for any $\boldsymbol{x} \neq \mathbf{0}$. □

- *$\boldsymbol{A} \succeq \mathbf{0}$ if and only if all its eigenvalues are nonnegative.*
- *$\boldsymbol{A} \prec \mathbf{0}$ if and only if all its eigenvalues are negative.*
- *$\boldsymbol{A} \preceq \mathbf{0}$ if and only if all its eigenvalues are nonpositive.*
- *$\boldsymbol{A}$ is indefinite if and only if it has at least one positive eigenvalue and at least one negative eigenvalue.*

Trace and determinant

- *If $\boldsymbol{A} \succ \mathbf{0} (\succeq \mathbf{0})$, then $Tr(\boldsymbol{A}) > (\geq) 0$ and $det(\boldsymbol{A}) > (\geq) 0$.*

  Idea of the proof: The trace and determinant of a symmetric matrix are the sum and product of its eigenvalues respectively. □

- *Above two conditions are necessary and sufficient for 2×2 matrix $\boldsymbol{A}$.*

  Idea of the proof: For any two real number $a, b \in \mathbb{R}$, one has $a, b > (\geq) 0$ if and only if $a + b > (\geq) 0$ and $ab > (\geq) 0$. □

- **Example:** Consider the matrices

$$\boldsymbol{A} := \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \boldsymbol{B} := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0.1 \end{bmatrix}.$$

(1) $\boldsymbol{A} \succ \mathbf{0}$ since $Tr(\boldsymbol{A}) = 7 > 0$ and $det(\boldsymbol{A}) = 11 > 0$.

(2) As for the matrix $\boldsymbol{B}$, $Tr(\boldsymbol{B}) = 2.1 > 0$ and $det(\boldsymbol{B}) = 0$. Even so, we cannot conclude that the matrix $\boldsymbol{B}$ is positive semidefinite. *In fact, $\boldsymbol{B}$ is indefinite since*

$$\boldsymbol{e}_1^\top \boldsymbol{B} \boldsymbol{e}_1 = 1 > 0, \quad (\boldsymbol{e}_2 - \boldsymbol{e}_3)^\top \boldsymbol{B} (\boldsymbol{e}_2 - \boldsymbol{e}_3) = -0.9 < 0.$$

Positive semidefinite square root

Given $\boldsymbol{A} \succeq \boldsymbol{0}$, let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\top$ be the spectral decomposition, where $\boldsymbol{U}$ is an orthogonal matrix, $\boldsymbol{D} = diag(d_1, d_2, \cdots, d_n)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\boldsymbol{A}$. Since $\boldsymbol{A} \succeq \boldsymbol{0}$, we have $d_1, d_2, \cdots, d_n \geq 0$. We define

$$\boldsymbol{A}^{\frac{1}{2}} := \boldsymbol{U}\boldsymbol{E}\boldsymbol{U}^\top, \quad \boldsymbol{E} = diag(\sqrt{d_1}, \sqrt{d_2}, \cdots, \sqrt{d_n}).$$

Obviously,

$$\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}} = \boldsymbol{U}\boldsymbol{E}\boldsymbol{U}^\top\boldsymbol{U}\boldsymbol{E}\boldsymbol{U}^\top = \boldsymbol{U}\boldsymbol{E}\boldsymbol{E}\boldsymbol{U}^\top = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\top = \boldsymbol{A}.$$

The matrix $\boldsymbol{A}^{\frac{1}{2}}$ is called the positive semidefinite square root.

Principal minors criterion

- **Definition:** *Given an $n \times n$ matrix, the determinant of the upper left $k \times k$ submatrix is called the kth principal minor, denoted by $D_k(\boldsymbol{A})$.*
- **Example:** The principal minors of the $k \times k$ matrix

$$\boldsymbol{A} := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

are $D_1(\boldsymbol{A}) = a_{11}$,

$$D_2(\boldsymbol{A}) = det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; D_3(\boldsymbol{A}) = det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

- **Principal minors criterion:** *Let $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then $\boldsymbol{A} \succ \mathbf{0}$ if and only if $D_1(\boldsymbol{A}) > 0, D_2(\boldsymbol{A}) > 0, \cdots, D_n(\boldsymbol{A}) > 0$.*

  Note: *It cannot be used for detecting positive semidefiniteness!*

Diagonally dominant matrices

- **Definition:** *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix. Then*
  (1) A *is called diagonally dominant if* $|A_{ii}| \geq \sum_{j\neq i} |A_{ij}|, \forall\, i$.
  (2) A *is called strictly diagonally dominant if* $|A_{ii}| > \sum_{j\neq i} |A_{ij}|, \forall\, i$.
- **Theorem:** *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix.*
  (1) *If* A *is a diagonally dominant matrix whose diagonal elements are nonnegative. Then* $A \succeq \mathbf{0}$.
  (2) *If* A *is a strictly diagonally dominant matrix whose diagonal elements are positive. Then* $A \succ \mathbf{0}$.

Proof: (1). Suppose $\exists\, \lambda < 0$ an eigenvalue of A. Let $u = (u_1, \cdots, u_n)^\top$ be a corresponding eigenvector. Let $|u_i| = \max\{|u_1|, \cdots, |u_n|\}$. Then by $Au = \lambda u$,

$$|A_{ii} - \lambda||u_i| = \left|\sum_{j\neq i} A_{ij}u_j\right| \leq \left(\sum_{j\neq i} |A_{ij}|\right)|u_i| \leq |A_{ii}||u_i|,$$

implying $|A_{ii} - \lambda| \leq |A_{ii}|$. This is a contradiction.

(2). From (1), we know that $A \succeq \mathbf{0}$. Thus, all we need to show is that A has no zero eigenvalues. Suppose $\exists$ eigenvalue $\lambda = 0$, $u \neq \mathbf{0}$ such that $Au = \mathbf{0}$. Similar to part (1), we obtain

$$|A_{ii}||u_i| = \left|\sum_{j\neq i} A_{ij}u_j\right| \leq \left(\sum_{j\neq i} |A_{ij}|\right)|u_i| < |A_{ii}||u_i|.$$

This is obviously a contradiction. □

Necessary second order optimality condition

Theorem: *Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Assume that f is twice continuously differentiable over U and that $\boldsymbol{x}^*$ is a stationary point. Then the following hold:*

(1) *If $\boldsymbol{x}^*$ is a local minimum point of f over U, then $\nabla^2 f(\boldsymbol{x}^*) \succeq \mathbf{0}$.*

(2) *If $\boldsymbol{x}^*$ is a local maximum point of f over U, then $\nabla^2 f(\boldsymbol{x}^*) \preceq \mathbf{0}$.*

Proof: (1). Since $\boldsymbol{x}^*$ is a local minimum point, $\exists\, B(\boldsymbol{x}^*, r) \subseteq U$ for which $f(\boldsymbol{x}^*) \le f(\boldsymbol{x})$, $\forall\, \boldsymbol{x} \in B(\boldsymbol{x}^*, r)$. Let $\mathbf{0} \neq \boldsymbol{d} \in \mathbb{R}^n$. For any $0 < \alpha < \frac{r}{\|\boldsymbol{d}\|}$, we have $\boldsymbol{x}_\alpha^* := \boldsymbol{x}^* + \alpha \boldsymbol{d} \in B(\boldsymbol{x}^*, r)$ and $f(\boldsymbol{x}_\alpha^*) \ge f(\boldsymbol{x}^*)$. By the linear approximation theorem, $\exists\, \boldsymbol{z}_\alpha \in (\boldsymbol{x}^*, \boldsymbol{x}_\alpha^*)$ such that

$$f(\boldsymbol{x}_\alpha^*) - f(\boldsymbol{x}^*) = \underbrace{\nabla f(\boldsymbol{x}^*)^\top}_{\mathbf{0}}(\boldsymbol{x}_\alpha^* - \boldsymbol{x}^*) + \frac{1}{2}(\boldsymbol{x}_\alpha^* - \boldsymbol{x}^*)^\top \nabla^2 f(\boldsymbol{z}_\alpha)(\boldsymbol{x}_\alpha^* - \boldsymbol{x}^*) = \frac{\alpha^2}{2}\boldsymbol{d}^\top \nabla^2 f(\boldsymbol{z}_\alpha)\boldsymbol{d}.$$

Thus, $\boldsymbol{d}^\top \nabla^2 f(\boldsymbol{z}_\alpha)\boldsymbol{d} \ge 0$, $\forall\, \alpha \in (0, \frac{r}{\|\boldsymbol{d}\|})$. Using the fact that $\boldsymbol{z}_\alpha \to \boldsymbol{x}^*$ as $\alpha \to 0^+$, and the continuity of the Hessian, we obtain $\boldsymbol{d}^\top \nabla^2 f(\boldsymbol{x}^*)\boldsymbol{d} \ge 0$. We conclude that $\nabla^2 f(\boldsymbol{x}^*) \succeq \mathbf{0}$.

(2). Employing the result of part (1) on the function $-f$, we obtain (2). $\square$

Sufficient second order optimality condition

Theorem: *Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Assume that f is twice continuously differentiable over U and that $\boldsymbol{x}^*$ is a stationary point. Then the following hold:*

(1) *If $\nabla^2 f(\boldsymbol{x}^*) \succ \mathbf{0}$, then $\boldsymbol{x}^*$ is a strict local minimum point of f over U.*

(2) *If $\nabla^2 f(\boldsymbol{x}^*) \prec \mathbf{0}$, then $\boldsymbol{x}^*$ is a strict local maximum point of f over U.*

Proof: (1) Since the Hessian is continuous, it follows that there exists a ball $B(\boldsymbol{x}^*, r) \subseteq U$ s.t. $\nabla^2 f(\boldsymbol{x}) \succ \mathbf{0}, \forall\, \boldsymbol{x} \in B(\boldsymbol{x}^*, r)$ (*using the principal minors criterion on page 16*). By the linear approximation theorem, it follows that for any $\boldsymbol{x} \in B(\boldsymbol{x}^*, r)$, $\exists\, \boldsymbol{z}_{\boldsymbol{x}} \in (\boldsymbol{x}^*, \boldsymbol{x}) \subset B(\boldsymbol{x}^*, r)$ such that

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^\top \nabla^2 f(\boldsymbol{z}_{\boldsymbol{x}})(\boldsymbol{x} - \boldsymbol{x}^*).$$

Since $\nabla^2 f(\boldsymbol{z}_{\boldsymbol{x}}) \succ \mathbf{0}$, it follows that

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) > 0, \quad \text{for } \boldsymbol{x} \neq \boldsymbol{x}^*.$$

That is, $\boldsymbol{x}^*$ is a strict local minimum point of f over U.

(2) This part follows from part (1) by considering the function $-f$. $\square$

Sufficient condition for a saddle point

- **Definition:** *Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Assume that f is continuously differentiable over U. A stationary point $\boldsymbol{x}^*$ is called a saddle point of f over U if it is neither a local minimum point nor a local maximum point of f over U.*
- **Sufficient condition for a saddle point:** *Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Assume that f is twice continuously differentiable over U and that $\boldsymbol{x}^*$ is a stationary point. If $\nabla^2 f(\boldsymbol{x}^*)$ is an indefinite matrix, then $\boldsymbol{x}^*$ is a saddle point of f over U.*

  Proof: Let $\lambda > 0$ be an eigenvalue of $\nabla^2 f(\boldsymbol{x}^*)$ with a normalized eigenvector $\boldsymbol{v}$. Since U is open, $\exists\, r > 0$ such that $\boldsymbol{x}^* + \alpha\boldsymbol{v} \in U, \forall\, \alpha \in (0, r)$. By the quadratic approximation theorem and $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$, we have

$$\begin{aligned} f(\boldsymbol{x}^* + \alpha\boldsymbol{v}) &= f(\boldsymbol{x}^*) + \frac{\alpha^2}{2}\boldsymbol{v}^\top \nabla^2 f(\boldsymbol{x}^*)\boldsymbol{v} + o(\alpha^2\|\boldsymbol{v}\|^2) \\ &= f(\boldsymbol{x}^*) + \frac{\lambda\alpha^2}{2}\|\boldsymbol{v}\|^2 + o(\alpha^2\|\boldsymbol{v}\|^2) = f(\boldsymbol{x}^*) + \frac{\lambda\alpha^2}{2} + o(\alpha^2). \end{aligned}$$

  Since $\frac{o(\alpha^2)}{\alpha^2} \to 0$ as $\alpha \to 0^+$, $\exists\, \varepsilon_1 \in (0, r)$ such that $o(\alpha^2) > -\frac{\lambda}{2}\alpha^2, \forall\, \alpha \in (0, \varepsilon_1)$. Hence, $f(\boldsymbol{x}^* + \alpha\boldsymbol{v}) > f(\boldsymbol{x}^*)$. This shows that $\boldsymbol{x}^*$ cannot be a local maximum point of f over U. Similarly, we can show that $\boldsymbol{x}^*$ cannot be a local minimum point of f over U. Therefore, $\boldsymbol{x}^*$ is a saddle point of f over U. $\square$

Weierstrass theorem

- **Weierstrass Theorem:** *Let $f : \emptyset \neq C \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous function and C is a compact set. Then there exist a global minimum point of f over C and a global maximum point of f over C.*
- **Definition:** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined over $\mathbb{R}^n$. The function f is called coercive if* $\lim_{\|\boldsymbol{x}\|\to\infty} f(\boldsymbol{x}) = \infty$.
- **Attainment under coerciveness:** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and coercive function and let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then f has a global minimum point over S.*

  Proof:

  (1) Let $\boldsymbol{x}_0 \in S$. Since f is coercive, $\exists\, M > 0$ such that $f(\boldsymbol{x}) > f(\boldsymbol{x}_0)$, $\forall\, \boldsymbol{x} \in \mathbb{R}^n$ and $\|\boldsymbol{x}\| > M$.

  (2) Since any global minimizer x^* of f over S satisfies $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}_0)$, it follows that the set of global minimizers of f over S is the same as the set of global minimizers of f over $S \cap B[\mathbf{0}, M]$.

  (3) The set $S \cap B[\mathbf{0}, M]$ is compact and nonempty, by the Weierstrass theorem, there exists a global minimizer of f over $S \cap B[\mathbf{0}, M]$ and hence also over S.

  $\square$

Example 1

Consider $f(x_1, x_2) = x_1^2 + x_2^2$ over the set $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq -1\}$.

- Since C is not bounded, the Weierstrass theorem does not guarantee the existence of global minimizer and maximizer of f over C. Obviously, f has no global maximizer over C.
- f is coercive and C is closed, f has a global minimizer over C.
- In the interior of C: $\nabla f(x_1, x_2) = \mathbf{0} \Rightarrow (x_1, x_2) = (0, 0) \notin C$.
  At the boundary of C: $\{(x_1, x_2) : x_1 + x_2 = -1\} \Rightarrow x_1 = -x_2 - 1$.
  $g(x_2) := f(-x_2 - 1, x_2) = (-x_2 - 1)^2 + x_2^2$
  $g'(x_2) = 2(1 + x_2) + 2x_2 \Rightarrow g'(x_2) = 0 \Rightarrow x_2 = -\frac{1}{2} \Rightarrow x_1 = -\frac{1}{2}$.

  Thus, $(x_1, x_2) = (-\frac{1}{2}, -\frac{1}{2})$ is the only candidate for a global minimum point.
  Therefore, $(x_1, x_2) = (-\frac{1}{2}, -\frac{1}{2})$ is the global minimum point of f over C.

Example 2

Consider the function $f(x_1, x_2) = 2x_1^3 + 3x_2^2 + 3x_1^2 x_2 - 24x_2$ over $\mathbb{R}^2$.

- $\nabla f(x_1, x_2) = \begin{bmatrix} 6x_1^2 + 6x_1 x_2 \\ 6x_2 + 3x_1^2 - 24 \end{bmatrix} := \mathbf{0}$. Then the stationary points of the function f are $(x_1, x_2) = (0, 4), (4, -4), (-2, 2)$.
- The Hessian of f is given by $\nabla^2 f(x_1, x_2) = 6 \begin{bmatrix} 2x_1 + x_2 & x_1 \\ x_1 & 1 \end{bmatrix}$.
- $\nabla^2 f(0, 4) = 6 \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \succ \mathbf{0} \Rightarrow$ $(0, 4)$ *is a strict local minimum point. It is not a global minimum point, since* $f(x_1, 0) = 2x_1^3 \to -\infty$ *as* $x_1 \to -\infty$.

  $\nabla^2 f(4, -4) = 6 \begin{bmatrix} 4 & 4 \\ 4 & 1 \end{bmatrix}$, $tr(\mathbf{A}) > 0$ but $det(\mathbf{A}) < 0$, an indefinite matrix.

  $\therefore$ $(4, -4)$ *is a saddle point*

  $\nabla^2 f(-2, 2) = 6 \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$ is indefinite, since it has both positive and negative elements on its diagonal (cf. page 10).

  $\therefore$ $(-2, 2)$ *is a saddle point*

Example 3

Consider the function $f(x_1, x_2) = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 - 1)^2$ over $\mathbb{R}^2$.

- $\nabla f(x_1, x_2) = 4 \begin{bmatrix} (x_1^2 + x_2^2 - 1)x_1 \\ (x_1^2 + x_2^2 - 1)x_2 + (x_2^2 - 1)x_2 \end{bmatrix} := \mathbf{0}.$

  Then the stationary points are $(0,0), (1,0), (-1,0), (0,1), (0,-1)$.
- The Hessian of the function is

  $\nabla^2 f(x_1, x_2) = 4 \begin{bmatrix} 3x_1^2 + x_2^2 - 1 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 + 6x_2^2 - 2 \end{bmatrix}.$
- $\nabla^2 f(0,0) = 4 \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \prec \mathbf{0}.$

  $\therefore$ $(0,0)$ *is a strict local maximum point*

  (not global, $\because$ $f(x_1, 0) = (x_1^2 - 1)^2 + 1 \to \infty$)

  $\nabla^2 f(1,0) = \nabla^2 f(-1,0) = 4 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, indefinite matrix.

  $\therefore$ $(1,0), (-1,0)$ *saddle points*

  $\nabla^2 f(0,1) = \nabla^2 f(0,-1) = 4 \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \succeq \mathbf{0}$, no conclusion!

  $\because$ $f(0,1) = f(0,-1) = 0$ and f is bounded below by 0

  $\therefore$ $(0,1), (0,-1)$ *are global minimum points*

Contour and surface plots of Example 3

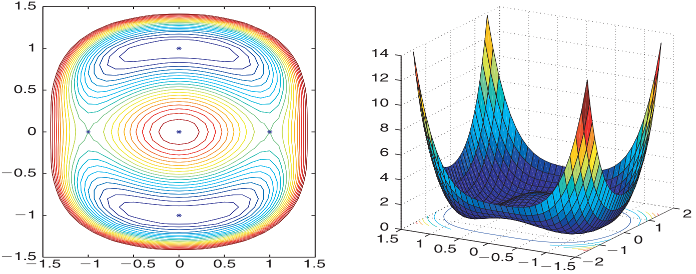


Figure 2.3. *Contour and surface plots of* $f(x_1, x_2) = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 - 1)^2$. *The five stationary points* $(0,0), (0,1), (0,-1), (1,0), (-1,0)$ *are denoted by asterisks. The points* $(0,-1), (0,1)$ *are strict local minimum points as well as global minimum points,* $(0,0)$ *is a local maximum point, and* $(-1,0), (1,0)$ *are saddle points.*

```
ezsurfc('(x^2 + y^2 -1)^2 + (y^2 - 1)^2', [-2 2 -1.5 1.5])
colorbar
view(-30, 30)
```

Example 4

Consider the function $f(x,y) = \dfrac{x+y}{x^2+y^2+1}$ over $\mathbb{R}^2$.

- $\nabla f(x,y) = \dfrac{1}{(x^2+y^2+1)^2}\begin{bmatrix}(x^2+y^2+1) - 2(x+y)x \\ (x^2+y^2+1) - 2(x+y)y\end{bmatrix} := \mathbf{0}. \Rightarrow$
  $-x^2 - 2xy + y^2 = -1,\ x^2 - 2xy - y^2 = -1$
  $\Rightarrow xy = 1/2$ (adding), $x^2 = y^2$ (subtracting)
  $\Rightarrow$ stationary points are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$
- For any $(x,y)^\top \in \mathbb{R}^2$, from the Cauchy-Schwarz inequality,

$$f(x,y) = \frac{(x,y)^\top \cdot (1,1)^\top}{x^2+y^2+1} \le \sqrt{2}\frac{\sqrt{x^2+y^2}}{x^2+y^2+1} \le \sqrt{2}\max_{t\ge 0}\frac{t}{t^2+1} \le \frac{\sqrt{2}}{2}.$$

  $\because\ (t-1)^2 \ge 0 \Rightarrow t^2+1 \ge 2t$
- $\because f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{\sqrt{2}}{2}$ $\quad\therefore$ $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *is the global maximum point*

  Similarly, $\because\ \dfrac{(-x,-y)^\top \cdot (1,1)^\top}{x^2+y^2+1} \le \dfrac{\sqrt{2}}{2}$ $\quad\therefore f(x,y) \ge \dfrac{-\sqrt{2}}{2}$

  $\because\ f(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = \frac{-\sqrt{2}}{2}$ $\quad\therefore$ $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ *is the global minimum point*

Example 5

Consider the function $f(x_1, x_2) = -2x_1^2 + x_1 x_2^2 + 4x_1^4$ over $\mathbb{R}^2$.

- $\nabla f(x_1, x_2) = \begin{bmatrix} -4x_1 + x_2^2 + 16x_1^3 \\ 2x_1 x_2 \end{bmatrix} := \mathbf{0}.$

  $\Rightarrow$ stationary points are $(0,0), (1/2,0), (-1/2,0)$.
- The Hessian of the function is $\nabla^2 f(x_1, x_2) = \begin{bmatrix} -4 + 48x_1^2 & 2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}$.
- $\nabla^2 f(1/2, 0) = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix} \succ \mathbf{0}.$ $\therefore$ $(1/2,0)$ *is a strict local minimum point (not global,* $f(-1, x_2) = 2 - x_2^2 \to -\infty,\ x_2 \to \infty$*)*

  $\nabla^2 f(-1/2, 0) = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}$, indefinite. $\therefore$ $(-1/2, 0)$ *saddle point*

  $\nabla^2 f(0,0) = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}$, a negative semidefinite matrix.

  $\because$ $f(\alpha^4, \alpha) = \alpha^6(-2\alpha^2 + 1 + 4\alpha^{10}) > 0$

  $f(-\alpha^4, \alpha) = \alpha^6(-2\alpha^2 - 1 + 4\alpha^{10}) < 0$ for $0 < \alpha \ll 1$

  $\therefore$ $(0,0)$ *is a saddle point of* f

Contour and surface plots of Example 5

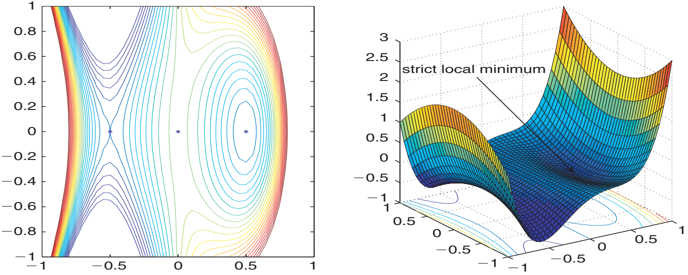


Figure 2.4. *Contour and surface plots of* $f(x_1, x_2) = -2x_1^2 + x_1x_2^2 + 4x_1^4$. *The three stationary point* $(0,0), (0.5,0), (-0.5,0)$ *are denoted by asterisks. The point* $(0.5,0)$ *is a strict local minimum, while* $(0,0)$ *and* $(-0.5,0)$ *are saddle points.*

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ezsurfc('-2*x^2 + x*y^2 + 4*x^4', [-1 1 -1 1])
colorbar
view(-45, 30)
```

Global optimality conditions

- **Theorem:** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Assume that $\nabla^2 f(\boldsymbol{x}) \succeq \boldsymbol{0}$, $\forall\, \boldsymbol{x} \in \mathbb{R}^n$. Let $\boldsymbol{x}^* \in \mathbb{R}^n$ be a stationary point of f. Then $\boldsymbol{x}^*$ is a global minimum point of f.*

  Proof: By the linear approximation theorem, $\forall\, \boldsymbol{x} \in \mathbb{R}^n$, $\exists\, \boldsymbol{z_x} \in (\boldsymbol{x}^*, \boldsymbol{x})$ such that

  $$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^\top \nabla^2 f(\boldsymbol{z_x})(\boldsymbol{x} - \boldsymbol{x}^*).$$

  Since $\nabla^2 f(\boldsymbol{z_x}) \succeq \boldsymbol{0}$, we have $f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)$. $\therefore\ \boldsymbol{x}^*$ is a global minimum point of f $\square$

- **Example:**

  $f(\boldsymbol{x}) := x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 + (x_1^2 + x_2^2 + x_3^2)^2.$

  $$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 + x_2 + x_3 + 4x_1(x_1^2 + x_2^2 + x_3^2) \\ 2x_2 + x_1 + x_3 + 4x_2(x_1^2 + x_2^2 + x_3^2) \\ 2x_3 + x_1 + x_2 + 4x_3(x_1^2 + x_2^2 + x_3^2) \end{bmatrix}.$$

  Obviously, $(x_1, x_2, x_3) = (0, 0, 0)$ is a stationary point.

  The Hessian is $\nabla^2 f(\boldsymbol{x}) = \boldsymbol{A} + \boldsymbol{B}(\boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x})$, where

  $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \succeq \boldsymbol{0}$, since it is diagnolly dominant with positive diagonal elements, $\boldsymbol{B}(\boldsymbol{x}) = 4(x_1^2 + x_2^2 + x_3^2)\boldsymbol{I}_3 \succeq \boldsymbol{0}$, and $\boldsymbol{C}(\boldsymbol{x}) = 8\boldsymbol{x}\boldsymbol{x}^\top \succeq \boldsymbol{0}$.

  $\therefore\ \nabla^2 f(\boldsymbol{x}) \succeq \boldsymbol{0}$ $\quad \therefore\ \boldsymbol{x} = (0, 0, 0)^\top$ *is a global minimum point of f over $\mathbb{R}^3$*

Quadratic functions

Quadratic functions are an important class of functions that are useful in the modeling of many optimization problems.

- **Definition:** *A quadratic function over* $\mathbb{R}^n$ *is a function of the form*

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c,$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ *is symmetric,* $\boldsymbol{b} \in \mathbb{R}^n$*, and* $c \in \mathbb{R}$.

- *The gradient and Hessian of the above quadratic function* $f(\boldsymbol{x})$*:*

$$\nabla f(\boldsymbol{x}) = 2\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}, \quad \nabla^2 f(\boldsymbol{x}) = 2\boldsymbol{A}.$$

- **Properties of quadratic functions:**

(1) $\boldsymbol{x}$ *is a stationary point of* f *iff* $\boldsymbol{A}\boldsymbol{x} = -\boldsymbol{b}$.

(2) *If* $\boldsymbol{A} \succeq \boldsymbol{0}$*, then* $\boldsymbol{x}$ *is a global minimum point of* f *iff* $\boldsymbol{A}\boldsymbol{x} = -\boldsymbol{b}$.

Proof: By Theorems on page 8 and page 29. □

(3) *If* $\boldsymbol{A} \succ \boldsymbol{0}$, $\boldsymbol{x} = -\boldsymbol{A}^{-1}\boldsymbol{b}$ *is a strict global minimum point of* f.

Proof: If $\boldsymbol{A} \succ \boldsymbol{0}$, then $\boldsymbol{x} = -\boldsymbol{A}^{-1}\boldsymbol{b}$ is the unique solution to $\boldsymbol{A}\boldsymbol{x} = -\boldsymbol{b}$. Hence, it is the unique global minimum point of f. □

Note: In (3), the minimal value of f is given by

$$f(\boldsymbol{x}) = (-\boldsymbol{A}^{-1}\boldsymbol{b})^\top \boldsymbol{A}(-\boldsymbol{A}^{-1}\boldsymbol{b}) - 2\boldsymbol{b}^\top \boldsymbol{A}^{-1}\boldsymbol{b} + c = c - \boldsymbol{b}^\top \boldsymbol{A}^{-1}\boldsymbol{b}.$$

Coerciveness of quadratic functions

Theorem: *Let $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\boldsymbol{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then f is coercive if and only if $\boldsymbol{A} \succ \boldsymbol{0}$.*

Proof:
(⇒) Assume that $\boldsymbol{A} \succ \boldsymbol{0}$. Then $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} \geq \alpha\|\boldsymbol{x}\|^2$ with $\alpha = \lambda_{\min}(\boldsymbol{A}) > 0$. Thus,

$$f(\boldsymbol{x}) \geq \alpha\|\boldsymbol{x}\|^2 - 2\|\boldsymbol{b}\|\|\boldsymbol{x}\| + c = \alpha\|\boldsymbol{x}\|\left(\|\boldsymbol{x}\| - 2\frac{\|\boldsymbol{b}\|}{\alpha}\right) + c \to \infty, \quad \text{as } \|\boldsymbol{x}\| \to \infty.$$

Therefore, f is coercive.
(⇐) Assume that f is coercive. We need to prove that $\boldsymbol{A} \succ \boldsymbol{0}$. We first show that there does not exist a negative eigenvalue. Suppose $\exists\, \boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n$, $\lambda < 0$ s.t. $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$. Then for any $\alpha \in \mathbb{R}$,

$$f(\alpha\boldsymbol{v}) = \lambda\|\boldsymbol{v}\|^2\alpha^2 + 2(\boldsymbol{b}^\top\boldsymbol{v})\alpha + c \to -\infty \quad \text{as } \alpha \to \infty.$$

This is a contradiction. We now show that 0 cannot be an eigenvalue of $\boldsymbol{A}$. Suppose $\exists\, \boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n$ s.t. $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$. Then for any $\alpha \in \mathbb{R}$,

$$f(\alpha\boldsymbol{v}) = 2(\boldsymbol{b}^\top\boldsymbol{v})\alpha + c.$$

If $\boldsymbol{b}^\top\boldsymbol{v} = 0$ then $f(\alpha\boldsymbol{v}) \to c$ as $\alpha \to \infty$. If $\boldsymbol{b}^\top\boldsymbol{v} > 0$ then $f(\alpha\boldsymbol{v}) \to -\infty$ as $\alpha \to -\infty$. If $\boldsymbol{b}^\top\boldsymbol{v} < 0$ then $f(\alpha\boldsymbol{v}) \to -\infty$ as $\alpha \to \infty$. All these contradict the coerciveness of f. □

Characterization of the nonnegativity of quadratic functions

Theorem: *Let $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\boldsymbol{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then the following two claims are equivalent:*

(a) $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c \geq 0, \forall\, \boldsymbol{x} \in \mathbb{R}^n$.

(b) $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^\top & c \end{bmatrix} \succeq \boldsymbol{0}$.

Proof:

(b) $\Rightarrow$ (a): For any $\boldsymbol{x} \in \mathbb{R}^n$, $0 \leq \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^\top & c \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ 1 \end{bmatrix} = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^\top \boldsymbol{x} + c \Rightarrow$ (a).

(a) $\Rightarrow$ (b): We begin by showing that $\boldsymbol{A} \succeq \boldsymbol{0}$.
Suppose not. $\exists\, \boldsymbol{0} \neq \boldsymbol{v} \in \mathbb{R}^n$ and $\lambda < 0$ s.t. $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$. Thus, for any $\alpha \in \mathbb{R}$,

$$f(\alpha \boldsymbol{v}) = \lambda \|\boldsymbol{v}\|^2 \alpha^2 + 2(\boldsymbol{b}^\top \boldsymbol{v})\alpha + c \to -\infty \quad \text{as } \alpha \to -\infty,$$

contradicting the nonnegativity of f. Our objective is to prove (b). We want to show that for any $\boldsymbol{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\begin{bmatrix} \boldsymbol{y} \\ t \end{bmatrix}^\top \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}^\top & c \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ t \end{bmatrix} \geq 0$, which is equivalent to

$$\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y} + 2t\boldsymbol{b}^\top \boldsymbol{y} + ct^2 \geq 0. \qquad (\star)$$

If $t = 0$ then $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y} + 2t\boldsymbol{b}^\top \boldsymbol{y} + ct^2 = \boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y} \geq 0$, since $\boldsymbol{A} \succeq \boldsymbol{0}$. We obtain $(\star)$.
If $t \neq 0$ then $0 \leq t^2 f(\boldsymbol{y}/t) = \boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y} + 2t\boldsymbol{b}^\top \boldsymbol{y} + ct^2$, we have $(\star)$. $\square$