

MA3113: Topics in Mathematical Image Processing I

ROF Model and Split Bregman Iterations



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The bounded variation space $BV(\Omega)$

Let Ω be an open subset of \mathbb{R}^2 . The space of functions of bounded variation $BV(\Omega)$ is defined as the space of real-valued function $u \in L^1(\Omega)$ such that the total variation is finite, i.e.,

$$BV(\Omega) = \{u \in L^1(\Omega) : \|u\|_{TV(\Omega)} < \infty\},$$

where

$$\|u\|_{TV(\Omega)} := \sup \left\{ \int_{\Omega} u(\nabla \cdot \varphi) \, dx : \varphi \in C_c^1(\Omega, \mathbb{R}^2), \|\varphi\|_{(L^\infty(\Omega))^2} \leq 1 \right\},$$

$C_c^1(\Omega, \mathbb{R}^2)$ is the space of continuously differentiable vector functions with compact support in Ω , $L^1(\Omega)$ and $L^\infty(\Omega)$ are the usual $L^p(\Omega)$ space for $p = 1$ and $p = \infty$, respectively.

Then $BV(\Omega)$ is a Banach space with the norm,

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|u\|_{TV(\Omega)}.$$

The ROF total-variation regularization model

Let $f : \overline{\Omega} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a given noisy image. Rudin, Osher, and Fatemi (*Physica D*, 1992) proposed the model for image denoising:

$$\min_{u \in BV(\Omega) \cap L^2(\Omega)} \left(\underbrace{\|u\|_{TV(\Omega)}}_{\text{regularizer}} + \frac{\lambda}{2} \underbrace{\int_{\Omega} (u(x) - f(x))^2 dx}_{\text{data fidelity}} \right),$$

where $\lambda > 0$ is a tuning parameter which controls the regularization strength. Notice that

- A smaller value of λ will lead to a more regular solution.
- The space of functions with bounded variation help remove spurious oscillations (noise) and preserve sharp signals (edges).
- The TV term allows the solution to have discontinuities.

The existence, uniqueness and stability of solution

Theorem: *If u is smooth, then $\|u\|_{TV(\Omega)} = \int_{\Omega} |\nabla u| dx$.*

Theorem: *If $f \in L^2(\Omega)$, the minimizer exists and is unique and is stable in L^2 with respect to perturbations in f .*

ROF model for image denoising: Below we assume that u is smooth, and we consider the model

$$\min_{u \in \mathcal{V}} \left(\int_{\Omega} |\nabla u| dx + \frac{\lambda}{2} \int_{\Omega} (u(x) - f(x))^2 dx \right).$$

Let $E[\cdot]$ be the functional over the vector space \mathcal{V} ,

$$E[u] := \int_{\Omega} |\nabla u| dx + \frac{\lambda}{2} \int_{\Omega} (u(x) - f(x))^2 dx.$$

Numerical experiments (Cameraman)

original



noisy(psnr 23.3449)



ROF(psnr 27.3876)



ROF(psnr 29.3811)



ROF(psnr 29.0611)



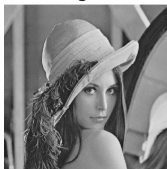
ROF(psnr 28.1453)



$$\lambda h = 1/5, 1/10, 1/15, 1/20$$

Numerical experiments (Lena)

original



noisy(psnr 23.0211)



ROF(psnr 27.538)



ROF(psnr 30.8209)



ROF(psnr 31.4951)



ROF(psnr 30.9829)



$$\lambda h = 1/5, 1/10, 1/15, 1/20$$

References

- ① L. I. Rudin, S. Osher, and E. Fatemi, Nonlinear total variation based noise removal algorithms, *Physica D*, 60 (1992), pp. 259-268.
- ② T. Goldstein and S. Osher, The split Bregman method for L^1 regularized problems, *SIAM Journal on Imaging Sciences*, 2 (2009), pp. 323-343.
- ③ P. Getreuer, Rudin-Osher-Fatemi total variation denoising using split Bregman, *Image Processing On Line*, 2 (2012), pp. 74-95.