MA3113: Topics in Mathematical Image Processing I
Variational Image Segmentation

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Outline of "variational image segmentation"

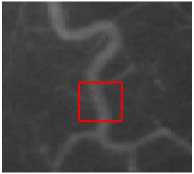
In this lecture, we will give a brief introduction to

- *the energy-based models for image segmentation: the Mumford-Shah model and the Chan-Vese model based on the level set formulation;*
- *an efficient iterative thresholding method for image segmentation; and*
- *a local intensity clustering model for intensity inhomogeneous images.*

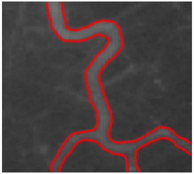
The material of this lecture is based on

- P. Getreuer, Chan-Vese segmentation, *Image Processing On Line,* 2 (2012), pp. 214-224.
- D. Wang, H. Li, X. Wei, X.-P. Wang (王筱平), An efficient iterative thresholding method for image segmentation, *Journal of Computational Physics,* 350 (2017), pp. 657-667.
- C. Li (李純明), R. Huang, Z. Ding, J. C. Gatenby, D. N. Metaxas, and J. C. Gore, A level set method for image segmentation in the presence of intensity inhomogeneities with application to MRI, *IEEE Transactions on Image Processing,* 20 (2011), pp. 2007-2016.

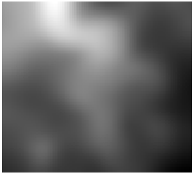
Image segmentation in medical imaging



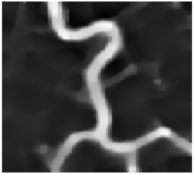
f & initialization C



segmented image



bias field b



corrected image I

Bias field model: $f = bI + n$, where n is the noise

In what follows, Ω denotes an open bounded subset in $\mathbb{R}^2$ and $f : \overline{\Omega} \to \mathbb{R}$ denotes the given grayscale image to be segmented.

Mumford-Shah model (CPAM 1989)

Mumford-Shah model: it finds a piecewise smooth function u and a curve set $\mathcal{C}$, which separates the image domain into disjoint regions, minimizing the energy functional:

$$\min_{u,\mathcal{C}} \left(\mu|\mathcal{C}| + \lambda \int_{\Omega} (f(\boldsymbol{x}) - u(\boldsymbol{x}))^2 \, d\boldsymbol{x} + \int_{\Omega \backslash \mathcal{C}} |\nabla u(\boldsymbol{x})|^2 \, d\boldsymbol{x} \right),$$

where $|\mathcal{C}|$ denotes the total length of the curves in $\mathcal{C}$.

- The first term plays the regularization role, which ensures the target objects can tightly be wrapped by $\mathcal{C}$.
- The second term is the data fidelity term, which forces u to be close to the input image f.
- The third term is the smoothing term, which forces the target function u to be piecewise smooth within each of the regions separated by the curves in $\mathcal{C}$.
- *$\mu > 0$, $\lambda > 0$ are tuning parameters to modulate these three terms.*

Simplified Mumford-Shah model

- *The non-convexity of energy functional in the Mumford-Shah model* makes the minimization problem difficult to analyze and the computational cost is much considerable.
- The piecewise smooth model suffers for its *sensitivity to the initialization of $\mathcal{C}$.*
- **Simplified Mumford-Shah model:** it finds *a piecewise constant function u* and a curve set $\mathcal{C}$ to minimize the energy functional:

$$\min_{u,\mathcal{C}} \left(\mu \left| \mathcal{C} \right| + \int_{\Omega} \left(f(\boldsymbol{x}) - u(\boldsymbol{x}) \right)^2 d\boldsymbol{x} \right).$$

Note that u is constant on each connected component of $\Omega \setminus \mathcal{C}$.
The minimization problem is still non-convex.

Chan-Vese two-phase model

In 1999, Chan (陳繁昌) and Vese proposed a two-phase segmentation model based on the level set formulation ("active contours without edges", LNCS 1999):

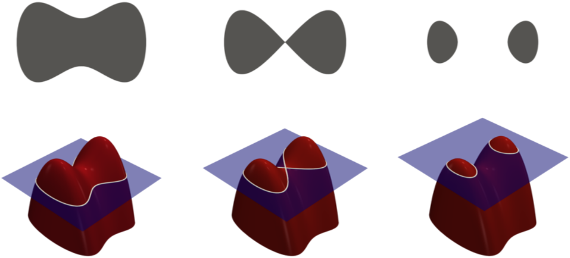
$$\min_{c_1,c_2,\mathcal{C}} \left(\mu|\mathcal{C}| + \nu|\Omega_{\text{in}}| + \lambda_1 \int_{\Omega_{\text{in}}} (f(\boldsymbol{x}) - c_1)^2 \, d\boldsymbol{x} + \lambda_2 \int_{\Omega_{\text{out}}} (f(\boldsymbol{x}) - c_2)^2 \, d\boldsymbol{x} \right).$$

- Ω_{in} denotes the region enclosed by the curves in $\mathcal{C}$ with area $|\Omega_{\text{in}}|$, and $\Omega_{\text{out}} := \Omega \setminus \Omega_{\text{in}}$.
- $\mu > 0$, $\nu \geq 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ are tuning parameters (actually, one of them can be fixed as 1).
- Chan-Vese model finds a piecewise constant function u and a curve set $\mathcal{C}$ to minimize the energy functional, where u has only two constant values,

$$u(\boldsymbol{x}) = \begin{cases} c_1, & \boldsymbol{x} \text{ is inside } \mathcal{C}, \\ c_2, & \boldsymbol{x} \text{ is outside } \mathcal{C}. \end{cases}$$

Topological changes of $\mathcal{C}$

To solve the minimization problem of Chan-Vese model, we evolve $\mathcal{C}$ and find c_1, c_2 to minimize the energy functional. However, it is generally hard to handle topological changes of the curves in $\mathcal{C}$.



(quoted from wikipedia)

Level set function

Therefore, we represent $\mathcal{C}$ implicitly by the zero level contour of a level set function $\phi : \overline{\Omega} \to \mathbb{R}$, i.e.,

$$\mathcal{C} = \{\boldsymbol{x} \in \overline{\Omega} : \ \phi(\boldsymbol{x}) = 0\}.$$

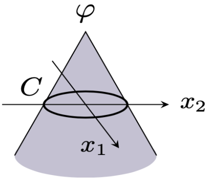
The zero level contour $\mathcal{C}$ partitions the image domain into two disjoint regions Ω_{in} and Ω_{out} such that

$$\phi(\boldsymbol{x}) \geq 0 \ \text{ for } \ \boldsymbol{x} \in \Omega_{\text{in}} \quad \text{and} \quad \phi(\boldsymbol{x}) < 0 \ \text{ for } \ \boldsymbol{x} \in \Omega_{\text{out}}.$$

For example, given $r > 0$, we define a level set function

$$\phi(\boldsymbol{x}) = \phi(x, y) = r - \sqrt{x^2 + y^2},$$

whose zero level contour is the circle of radius $r > 0$.



Chan-Vese model

- Let H denote the Heaviside function and δ the Dirac delta function. Then

$$H(s) = \begin{cases} 1 & s \geq 0, \\ 0 & s < 0, \end{cases} \qquad \text{and} \qquad \frac{d}{ds}H(s) = \delta(s).$$

- In terms of H, δ, and the level set function ϕ, the Chan-Vese model has the form

$$\begin{aligned} \min_{c1,\, c2,\, \phi} \Big(\mu \int_\Omega \delta(\phi(\boldsymbol{x})) |\nabla \phi(\boldsymbol{x})| \, d\boldsymbol{x} + \nu \int_\Omega H(\phi(\boldsymbol{x})) \, d\boldsymbol{x} \\ + \lambda_1 \int_\Omega (f(\boldsymbol{x}) - c_1)^2 H(\phi(\boldsymbol{x})) \, d\boldsymbol{x} \\ + \lambda_2 \int_\Omega (f(\boldsymbol{x}) - c_2)^2 (1 - H(\phi(\boldsymbol{x}))) \, d\boldsymbol{x} \Big). \end{aligned}$$

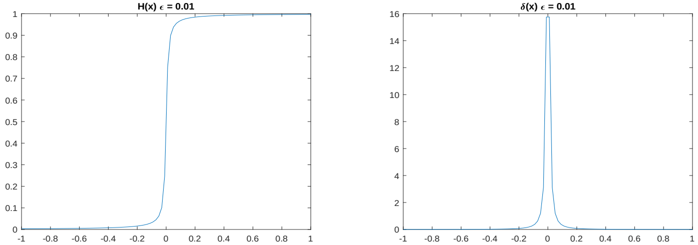
Original formulation:

$$\min_{c_1, c_2, \mathcal{C}} \left(\mu |\mathcal{C}| + \nu |\Omega_{\text{in}}| + \lambda_1 \int_{\Omega_{\text{in}}} (f(\boldsymbol{x}) - c_1)^2 + \lambda_2 \int_{\Omega_{\text{out}}} (f(\boldsymbol{x}) - c_2)^2 \right).$$

The regularized Heaviside and delta functions

The Heaviside function H and the Dirac delta function δ can be approximately regularized as follows: for a sufficiently small $\epsilon > 0$,

$$\begin{aligned} H_\epsilon(t) &:= \frac{1}{2}\Big(1 + \frac{2}{\pi}\tan^{-1}\big(\frac{t}{\epsilon}\big)\Big), \\ \delta_\epsilon(t) &:= \frac{d}{dt}H_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2 + t^2)}, \\ \int_{-\infty}^{\infty} \delta_\epsilon(t)dt &= \int_{-\infty}^{\infty} \frac{\epsilon}{\pi(\epsilon^2+t^2)}dt = \cdots = 1. \end{aligned}$$



Total length of $\mathcal{C}$

The first term of the energy functional is the length of $\mathcal{C}$, which can be expressed as the total variation of $H(\phi)$,

$$\begin{aligned} |\mathcal{C}| &= \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} = \int_\Omega \left|\frac{dH}{d\phi}(\phi(\boldsymbol{x}))\right||\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \\ &= \int_\Omega |\nabla H(\phi(\boldsymbol{x}))|d\boldsymbol{x}. \end{aligned}$$

A heuristic argument to prove $|\mathcal{C}| = \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x}$:

Suppose that the level set function ϕ is a signed distance function, i.e.,

$$\phi(\boldsymbol{x}) = \begin{cases} d(\boldsymbol{x},\mathcal{C}) & \text{if } \boldsymbol{x} \in \Omega_{\text{in}}, \\ -d(\boldsymbol{x},\mathcal{C}) & \text{if } \boldsymbol{x} \in \Omega_{\text{out}}. \end{cases}$$

Then $\phi(\boldsymbol{x})$ is differentiable almost everywhere and $|\nabla\phi(\boldsymbol{x})| = 1$ for $\boldsymbol{x} \in \mathbb{R}^2 \setminus \partial\Omega$ a.e. The contour $\mathcal{C}$ can be parametrized in arc length s, $\boldsymbol{z}(s) = (x(s), y(s))$ for $0 \le s \le L := |\mathcal{C}|$. Let $N \gg 1$ be a large integer. We approximate the Dirac δ-function by

$$\delta_N(t) := \begin{cases} N, & |t| \le \frac{1}{2N}, \\ 0, & \text{otherwise.} \end{cases}$$

Total length of $\mathcal{C}$: a heuristic argument (cont'd)

Let B_N be the narrow band defined by

$$B_N := \{\boldsymbol{x} \in \overline{\Omega} : \ |\phi(\boldsymbol{x})| \le 1/(2N)\}.$$

Then we have

$$\int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \approx N \int_{B_N} |\nabla\phi(\boldsymbol{x})|d\boldsymbol{x}.$$

The "centerline" of this band B_N is the curve $\mathcal{C} = \{\boldsymbol{x} \in \overline{\Omega} : \ \phi(\boldsymbol{x}) = 0\}$. Consider a point $\boldsymbol{p} = \boldsymbol{z}(s) \in \mathcal{C}$. Then the tangent vector and the normal vector are $\boldsymbol{z}'(s) = (x'(s), y'(s))$ and $\nabla\phi(\boldsymbol{z}(s))$, respectively. Starting at $\boldsymbol{p}$ in the direction $\nabla\phi(\boldsymbol{p})$, we reach the boundary of B_N when we have traversed the length $h > 0$ such that $|\nabla\phi(\boldsymbol{p})|h = \frac{1}{2N}$. It follows that near $\boldsymbol{p} = \boldsymbol{z}(s)$ the width $\rho(s)$ of this band is approximately given by

$$\rho(s) = 2h = \frac{1}{N|\nabla\phi(\boldsymbol{z}(s))|} = \frac{1}{N}.$$

Therefore we have

$$\int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \approx N \int_{B_N} |\nabla\phi(\boldsymbol{x})|d\boldsymbol{x} \approx N \int_0^L \rho(s)ds = L = |\mathcal{C}|.$$

An alternating iterative scheme

The minimization is solved by *an alternating iterative scheme*, i.e., alternatingly updating c_1, c_2 and ϕ.

(S1) Fixed ϕ, the optimal values of c_1 and c_2 are the region averages,

$$c_1 = \frac{\int_\Omega f(\boldsymbol{x}) H(\phi(\boldsymbol{x}))\, d\boldsymbol{x}}{\int_\Omega H(\phi(\boldsymbol{x}))\, d\boldsymbol{x}}, \quad c_2 = \frac{\int_\Omega f(\boldsymbol{x}) \big(1 - H(\phi(\boldsymbol{x}))\big)\, d\boldsymbol{x}}{\int_\Omega \big(1 - H(\phi(\boldsymbol{x}))\big)\, d\boldsymbol{x}}.$$

(S2) Fixed c_1, c_2, we solve the initial-boundary value problem (IBVP) to reach a steady-state:

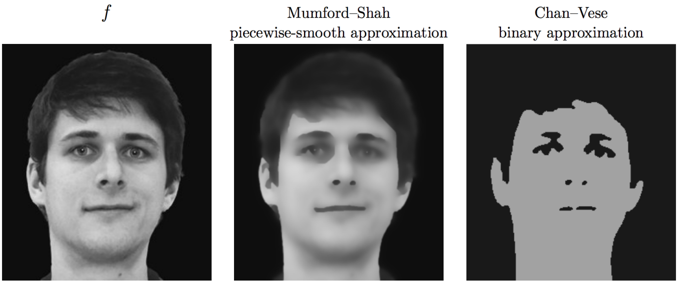
$$\frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi)\Big(\mu \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} - \nu - \lambda_1 (f - c_1)^2 + \lambda_2 (f - c_2)^2\Big),$$

$$\text{for } \ t > 0, \boldsymbol{x} \in \Omega,$$

$$\phi(0, \boldsymbol{x}) = \phi_0(\boldsymbol{x}), \boldsymbol{x} \in \Omega,$$

$$\frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega, t \geq 0.$$

Example: Mumford-Shah vs. Chan-Vese



P. Getreuer, Chan-Vese segmentation,
Image Processing On Line, 2 (2012), pp. 214-224.

Energy decreasing in time variable

The IBVP can also be derived by considering the decreasing of the Chan-Vese energy functional in time variable t.

(1) First, we introduce the time variable t and assume that the level set function ϕ evolves in time t, $\phi = \phi(t, x, y)$. Let $\Delta t > 0$ be an arbitrary small time step. *We suppose that the Chan-Vese energy functional is decreasing when the level set function ϕ evolves in time t.*

(2) For a given time $t \geq 0$, we define

$$\begin{aligned} v(x,y) &:= \frac{\partial \phi}{\partial t}(t,x,y)\Delta t \approx \phi(t+\Delta t, x, y) - \phi(t,x,y), \\ \psi(x,y) &:= \phi(t,x,y) + \alpha v(x,y) \approx \phi(t+\alpha\Delta t, x, y), \end{aligned}$$

where $0 < \alpha \ll 1$. Then $\psi_x = \phi_x + \alpha v_x$ and $\psi_y = \phi_y + \alpha v_y$.

(3) Let F be the integrand in the Chan-Vese energy functional. Then

$$\begin{aligned} E[\psi] &:= \int_\Omega F(x, y, \psi, \psi_x, \psi_y)\, d\mathbf{x}, \\ \left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} &= \int_\Omega \frac{\partial F}{\partial \phi} v + \frac{\partial F}{\partial \phi_x} v_x + \frac{\partial F}{\partial \phi_y} v_y \, d\mathbf{x} \leq 0. \end{aligned}$$

Energy decreasing in time variable (cont'd)

(4) Recall Green's formula,

$$\int_\Omega w \cdot \nabla p \, d\boldsymbol{x} = \int_{\partial\Omega} (w \cdot \boldsymbol{n}) p \, d\boldsymbol{\sigma} - \int_\Omega (\nabla \cdot w) p \, d\boldsymbol{x}.$$

Let $w = \left(\frac{\partial F}{\partial \phi_x}, \frac{\partial F}{\partial \phi_y}\right)$, $p = v$, and $\boldsymbol{n} = (n_1, n_2)$ be the unit normal vector to $\partial\Omega$. Then

$$\begin{aligned} \int_\Omega \frac{\partial F}{\partial \phi_x} v_x + \frac{\partial F}{\partial \phi_y} v_y \, d\boldsymbol{x} \quad &= \quad \int_{\partial\Omega} \left(\frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 \right) v \, d\boldsymbol{\sigma} \\ &\quad - \int_\Omega \left(\frac{\partial}{\partial x} \frac{\partial F}{\partial \phi_x} + \frac{\partial}{\partial y} \frac{\partial F}{\partial \phi_y} \right) v \, d\boldsymbol{x} \end{aligned}$$

Thus,

$$\begin{aligned} \left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} \quad &= \quad \int_\Omega \left\{ \frac{\partial F}{\partial \phi} v - \left(\frac{\partial}{\partial x} \frac{\partial F}{\partial \phi_x} \right) v - \left(\frac{\partial}{\partial y} \frac{\partial F}{\partial \phi_y} \right) v \right\} d\boldsymbol{x} \\ &\quad + \int_{\partial\Omega} \left(\frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 \right) v \, d\boldsymbol{\sigma}. \qquad (\star) \end{aligned}$$

Energy decreasing in time variable (cont'd)

Since

$$v(x,y) := \frac{\partial \phi}{\partial t}(t,x,y)\Delta t \approx \phi(t+\Delta t, x, y) - \phi(t,x,y),$$

it follows that $v(x,y) \approx 0$ for $(x,y) \in \partial\Omega$ and then

$$\left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} = \int_\Omega \left\{ \frac{\partial F}{\partial \phi} - \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial \phi_x}\right) - \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial \phi_y}\right) \right\} v\, d\boldsymbol{x} \leq 0.$$

Therefore, we obtain *a sufficient condition* for $\left.\frac{dE[\psi]}{d\alpha}\right|_{\alpha=0} \leq 0$,

$$\frac{\partial \phi}{\partial t}(t,x,y) = -\left\{ \frac{\partial F}{\partial \phi} - \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial \phi_x}\right) - \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial \phi_y}\right) \right\}.$$

Note that

$$\begin{aligned} F(x,y,\phi,\phi_x,\phi_y) &= \mu\delta_\epsilon(\phi)\,|\nabla\phi| + \nu H_\epsilon(\phi) + \lambda_1(f-c_1)^2 H_\epsilon(\phi) \\ &\quad + \lambda_2(f-c_2)^2(1-H_\epsilon(\phi)). \end{aligned}$$

Energy decreasing in time variable (cont'd)

By direct computations, we obtain

$$\begin{aligned}
\frac{\partial F}{\partial \phi} &= \mu \delta_\epsilon'(\phi)\,|\nabla \phi| + \nu \delta_\epsilon(\phi) + \lambda_1 (f - c_1)^2 \delta_\epsilon(\phi) - \lambda_2 (f - c_2)^2 \delta_\epsilon(\phi),\\
\frac{\partial F}{\partial \phi_x} &= \mu \delta_\epsilon(\phi) \frac{\phi_x}{\sqrt{\phi_x^2 + \phi_y^2}} = \mu \delta_\epsilon(\phi) \frac{\phi_x}{|\nabla \phi|},\\
\frac{\partial F}{\partial \phi_y} &= \mu \delta_\epsilon(\phi) \frac{\phi_y}{|\nabla \phi|}.
\end{aligned}$$

It leads to the equation

$$\frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi) \left\{ \mu \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right) - \nu - \lambda_1 (f - c_1)^2 + \lambda_2 (f - c_2)^2 \right\},$$

which has to be supplemented with an initial condition,

$$\phi(0, \boldsymbol{x}) = \phi_0(\boldsymbol{x}),\ \forall\, \boldsymbol{x} \in \Omega.$$

Neumann boundary condition

For a given time $t \geq 0$, if the energy functional E attains a local (or global) minimum at ϕ the we have

$$\int_{\partial\Omega} \left(\frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 \right) v \, d\boldsymbol{\sigma} = 0 \textit{ for any smooth function } v \textit{ on } \overline{\Omega}.$$

It follows that

$$0 = \frac{\partial F}{\partial \phi_x} n_1 + \frac{\partial F}{\partial \phi_y} n_2 = \left(\frac{\partial F}{\partial \phi_x}, \frac{\partial F}{\partial \phi_y} \right) \cdot \boldsymbol{n} = \delta_\epsilon(\phi) \frac{\nabla \phi}{|\nabla \phi|} \cdot \boldsymbol{n}.$$

That is, we obtain the BC for $t \geq 0$,

$$\frac{\delta_\epsilon(\phi)}{|\nabla \phi|} \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega \quad \Longrightarrow \quad \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega$$

Numerical implementation

- Assume that the image domain $\overline{\Omega}$ is the unit square $[0,1] \times [0,1]$.
- Let $\Omega_D := \{(x_i, y_j) | \ i, j = 0, 1, \cdots, M\}$ be the set of grid points of a uniform partition of $\overline{\Omega}$ with size $h = 1/M$.
- Then $x_i = ih$ and $y_j = jh$, $i, j = 0, 1, \cdots, M$. Let $\phi_{i,j}(t)$ be the spatial difference approximation to $\phi(t, x_i, y_j)$.
- Let $t_n = n\Delta t$, $n \geq 0$, and $\Delta t > 0$ be the time step, and let $\phi^n_{i,j}$ be the full difference approximation to $\phi(t_n, x_i, y_j)$.

Discrete differential operators and BC

- *Define the discrete differential operators: for $1 \leq i,j \leq M-1$,*

$$
\begin{aligned}
\nabla_x^+ \phi_{i,j} &= \frac{\phi_{i+1,j} - \phi_{i,j}}{h}, \text{ (forward difference)} \\
\nabla_x^- \phi_{i,j} &= \frac{\phi_{i,j} - \phi_{i-1,j}}{h}, \text{ (backward difference)} \\
\nabla_y^+ \phi_{i,j} &= \frac{\phi_{i,j+1} - \phi_{i,j}}{h}, \text{ (forward difference)} \\
\nabla_y^- \phi_{i,j} &= \frac{\phi_{i,j} - \phi_{i,j-1}}{h}, \text{ (backward difference)} \\
\nabla_x^0 \phi_{i,j} &:= \left(\frac{\nabla_x^+ + \nabla_x^-}{2}\right)\phi_{i,j}, \quad \nabla_y^0 \phi_{i,j} := \left(\frac{\nabla_y^+ + \nabla_y^-}{2}\right)\phi_{i,j}. \\
& \text{(central differences)}
\end{aligned}
$$

- *Discretize the homogeneous Neumann BC:* $\dfrac{\partial \phi}{\partial \boldsymbol{n}} = 0$ *on* $\partial\Omega$

$$
\phi_{0,j} = \phi_{1,j}, \quad \phi_{M,j} = \phi_{M-1,j}, \quad \phi_{i,0} = \phi_{i,1}, \quad \phi_{i,M} = \phi_{i,M-1}.
$$

Finite difference discretization: spatial variables

Performing the spatial discretization [Getreuer-2012], we have

$$
\begin{aligned}
\frac{\partial \phi_{i,j}}{\partial t} = \delta_\epsilon(\phi_{i,j})\Bigg\{ & \mu\Bigg(\nabla_x^- \frac{\nabla_x^+ \phi_{i,j}}{\sqrt{\eta^2 + (\nabla_x^+ \phi_{i,j})^2 + (\nabla_y^0 \phi_{i,j})^2}} \\
& + \nabla_y^- \frac{\nabla_y^+ \phi_{i,j}}{\sqrt{\eta^2 + (\nabla_x^0 \phi_{i,j})^2 + (\nabla_y^+ \phi_{i,j})^2}}\Bigg) \\
& - \nu - \lambda_1 (f_{i,j} - c_1)^2 + \lambda_2 (f_{i,j} - c_2)^2 \Bigg\},
\end{aligned}
$$

where $i, j = 1, 2, \cdots, M-1$.

The purpose of small positive parameter η in the denominators prevents division by zero.

Spatial discretization

Define

$$A_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^+ \phi_{i,j})^2 + (\nabla_y^0 \phi_{i,j})^2}},$$

$$B_{i,j} = \frac{\mu}{\sqrt{\eta^2 + (\nabla_x^0 \phi_{i,j})^2 + (\nabla_y^+ \phi_{i,j})^2}}.$$

Using the fact $\nabla_x^+ \phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{h}$, $\nabla_y^+ \phi_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{h}$ and taking the backward difference at $A_{i,j}(\phi_{i+1,j} - \phi_{i,j})$ and $B_{i,j}(\phi_{i,j+1} - \phi_{i,j})$, then the discretization can be written as

$$\begin{aligned} \frac{\partial \phi_{i,j}}{\partial t} = \delta_\epsilon(\phi_{i,j}) \Bigg\{ & \frac{1}{h^2}\Big(A_{i,j}(\phi_{i+1,j} - \phi_{i,j}) - A_{i-1,j}(\phi_{i,j} - \phi_{i-1,j}) \Big) \\ & + \frac{1}{h^2}\Big(B_{i,j}(\phi_{i,j+1} - \phi_{i,j}) - B_{i,j-1}(\phi_{i,j} - \phi_{i,j-1}) \Big) \\ & - \nu - \lambda_1 (f_{i,j} - c_1)^2 + \lambda_2 (f_{i,j} - c_2)^2 \Bigg\}. \end{aligned}$$

Temporal discretization

Define

$$\widetilde{A}_{i,j} = \frac{1}{h^2} A_{i,j}, \quad \widetilde{A}_{i-1,j} = \frac{1}{h^2} A_{i,j},$$
$$\widetilde{B}_{i,j} = \frac{1}{h^2} B_{i,j}, \quad \widetilde{B}_{i,j-1} = \frac{1}{h^2} B_{i,j-1}.$$

Time is discretized with a semi-implicit Gauss-Seidel method, values $\phi_{i,j}$, $\phi_{i-1,j}$, $\phi_{i,j-1}$ are evaluated at time t_{n+1} and all others at time t_n.

$$\begin{aligned}\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t} \quad = \quad & \delta_\epsilon(\phi_{i,j}^n) \Big\{ \widetilde{A}_{i,j}\phi_{i+1,j}^{n} + \widetilde{A}_{i-1,j}\phi_{i-1,j}^{n+1} + \widetilde{B}_{i,j}\phi_{i,j+1}^{n} + \widetilde{B}_{i,j-1}\phi_{i,j-1}^{n+1} \\ & - \left(\widetilde{A}_{i,j} + \widetilde{A}_{i-1,j} + \widetilde{B}_{i,j} + \widetilde{B}_{i,j-1} \right) \phi_{i,j}^{n+1} \\ & - \nu - \lambda_1 (f_{i,j} - c_1)^2 + \lambda_2 (f_{i,j} - c_2)^2 \Big\}.\end{aligned}$$

Gauss-Seidel scheme

This allows ϕ at time t_{n+1} to be solved by one Gauss-Seidel *sweep from left to right, bottom to top:*

$$
\begin{aligned}
\phi_{i,j}^{n+1} \quad = \quad & \Big\{ \phi_{i,j}^{n} + \Delta t \delta_\epsilon(\phi_{i,j}^{n}) \Big(\widetilde{A}_{i,j}\phi_{i+1,j}^{n} + \widetilde{A}_{i-1,j}\phi_{i-1,j}^{n+1} + \widetilde{B}_{i,j}\phi_{i,j+1}^{n} \\
& + \widetilde{B}_{i,j-1}\phi_{i,j-1}^{n+1} - \nu - \lambda_1 (f_{i,j} - c_1)^2 + \lambda_2 (f_{i,j} - c_2)^2 \Big) \Big\} \\
& \times \Big\{ 1 + \Delta t \delta_\epsilon(\phi_{i,j}) \Big(\widetilde{A}_{i,j} + \widetilde{A}_{i-1,j} + \widetilde{B}_{i,j} + \widetilde{B}_{i,j-1} \Big) \Big\}^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{A}_{i,j} &= \frac{\mu}{h^2 \sqrt{\eta^2 + \Big((\phi_{i+1,j}^{n} - \phi_{i,j}^{n})/h \Big)^2 + \Big((\phi_{i,j+1}^{n} - \phi_{i,j-1}^{n+1})/(2h) \Big)^2}}, \\
\widetilde{B}_{i,j} &= \frac{\mu}{h^2 \sqrt{\eta^2 + \Big((\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n+1})/(2h) \Big)^2 + \Big((\phi_{i,j}^{n} - \phi_{i+1,j}^{n})/h \Big)^2}}.
\end{aligned}
$$

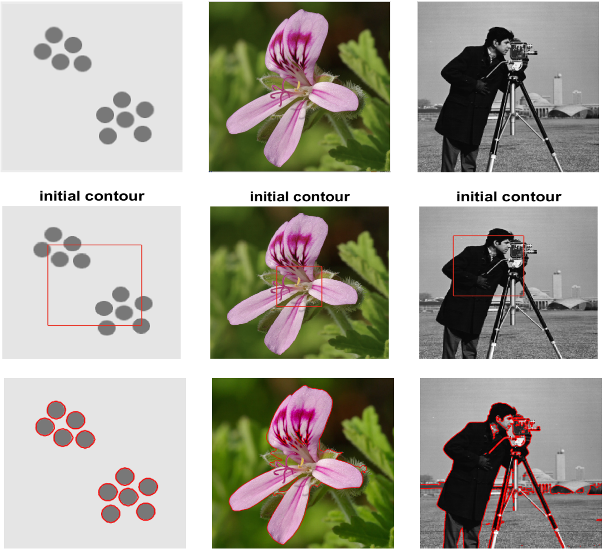
Gauss-Seidel scheme

We can rewrite $\widetilde{A}_{i,j}$ and $\widetilde{B}_{i,j}$ as follows:

$$\begin{aligned}
\widetilde{A}_{i,j} &= \frac{\mu}{h^2\sqrt{\eta^2 + \left((\phi^n_{i+1,j} - \phi^n_{i,j})/h\right)^2 + \left((\phi^n_{i,j+1} - \phi^{n+1}_{i,j-1})/(2h)\right)^2}}, \\
&= \frac{(\mu/h)}{\sqrt{(h\eta)^2 + (\phi^n_{i+1,j} - \phi^n_{i,j})^2 + \left((\phi^n_{i,j+1} - \phi^{n+1}_{i,j-1})/2\right)^2}}, \\
\widetilde{B}_{i,j} &= \frac{\mu}{h^2\sqrt{\eta^2 + \left((\phi^n_{i+1,j} - \phi^{n+1}_{i-1,j})/(2h)\right)^2 + \left((\phi^n_{i,j} - \phi^n_{i+1,j})/h\right)^2}} \\
&= \frac{(\mu/h)}{\sqrt{(h\eta)^2 + \left((\phi^n_{i+1,j} - \phi^{n+1}_{i-1,j})/2\right)^2 + (\phi^n_{i,j} - \phi^n_{i+1,j})^2}}.
\end{aligned}$$

In numerical implementation, we take $(h\eta) = 10^{-8}$.

Numerical experiments



The iterative convolution-thresholding scheme

- Most image segmentation models incorporate the level set formulation for solving the associated minimization problems. It generally results in initial-boundary value problems for PDEs.
- We are going to employ an *iterative convolution-thresholding (ICT) scheme* [WLWW-JCP2017] for multi-phase image segmentation based on the Chan-Vese model.
- In the ICT scheme, total length of $\mathcal{C}$ is approximated by a non-local multi-phase energy constructed based on *convolution of the heat kernel with the characteristic functions of regions.*
- The ICT scheme is divided into two steps. *It works by alternating a convolution step with the thresholding step.* The convolution can be implemented efficiently on a uniform mesh using the fast Fourier transform (FFT) with the optimal complexity of $O(N \log N)$ per iteration.

The approximate Chan-Vese functional

Let $f : \overline{\Omega} \to \mathbb{R}$ be the given grayscale image to be segmented.

- Suppose f approximately takes n distinct constants $c_1, \cdots, c_n$ in the disjoint regions $\Omega_1, \cdots, \Omega_n$ *(n-phase partition)* with boundaries $\mathcal{C}_1, \cdots, \mathcal{C}_n$, respectively, that separate Ω.

  Let $\mathcal{C} = \cup_{i=1}^n \mathcal{C}_i$. Then $\Omega \setminus \mathcal{C} = \cup_{i=1}^n \Omega_i$.
- Let χ_i be the characteristic function of the desirable region Ω_i,

$$\chi_i(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Omega_i, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \sum_{i=1}^n \chi_i = 1 \text{ in } \Omega \setminus \mathcal{C}.$$

- Let $\boldsymbol{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$. We define the set $\mathcal{S}$ of the characteristic vector functions by

$$\mathcal{S} = \left\{ \boldsymbol{\chi} \in (BV(\Omega))^n : \chi_i(\boldsymbol{x}) \in \{0,1\}, \sum_{i=1}^n \chi_i(\boldsymbol{x}) = 1 \ \forall \boldsymbol{x} \in \Omega \setminus \mathcal{C} \right\},$$

  where $BV(\Omega)$ is the usual bounded variation space.

The approximate Chan-Vese functional (cont'd)

In [WLWW-JCP2017], the authors considered the following model:

$$\min_{\{\Omega_i\},\{c_i\}} \sum_{i=1}^{n} \left(\lambda |\mathcal{C}_i| + \int_{\Omega_i} (f(\boldsymbol{x}) - c_i)^2 \, d\boldsymbol{x} \right).$$

Let $\boldsymbol{c} := (c_1, c_2, \cdots, c_n)$. Then we look for $\boldsymbol{\chi}^*$ and $\boldsymbol{c}^*$ such that

$$(\boldsymbol{\chi}^*, \boldsymbol{c}^*) = \arg\min_{\boldsymbol{\chi}\in\mathcal{S}, \boldsymbol{c}\in\mathbb{R}^n} \sum_{i=1}^{n} \left(\lambda |\mathcal{C}_i| + \int_{\Omega} \chi_i(\boldsymbol{x}) g_i(\boldsymbol{x}) \, d\boldsymbol{x} \right),$$

where

$$g_i(\boldsymbol{x}) := (f(\boldsymbol{x}) - c_i)^2.$$

The length of $\mathcal{C}_i$

Let $0 < \tau \ll 1$. Define the heat kernel G_τ by

$$G_\tau(\boldsymbol{x}) := \frac{1}{4\pi\tau} \exp\Big(-\frac{\|\boldsymbol{x}\|_2^2}{4\tau}\Big).$$

Then the length of $\mathcal{C}_i \cap \mathcal{C}_j$ can be approximated by (see CPAM-2015)

$$|\mathcal{C}_i \cap \mathcal{C}_j| \approx \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x},$$

where $*$ represents the convolution operation, and therefore

$$|\mathcal{C}_i| \approx \sum_{j=1, j\neq i}^{n} \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x}.$$

S. Esedoḡlu and F. Otto, Threshold dynamics for networks with arbitrary surface tensions, *Communications on Pure and Applied Mathematics*, 68 (2015), pp. 808-864.

The approximate energy functional and ICT scheme

The total energy functional can be approximated by

$$\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}) = \sum_{i=1}^{n} \Big(\lambda \sum_{j=1, j\neq i}^{n} \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \chi_i(\boldsymbol{x}) g_i(\boldsymbol{x})\, d\boldsymbol{x} \Big),$$

and our goal is to solve the following minimization problem:

$$(\boldsymbol{\chi}^*, \boldsymbol{c}^*) = \underset{\boldsymbol{\chi}\in\mathcal{S}, \boldsymbol{c}\in\mathbb{R}^n}{\arg\min}\ \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}).$$

The minimization problem can be solved by the ICT scheme, i.e., alternatively updating $\boldsymbol{\chi}$ and $\boldsymbol{c}$. Suppose that we have the k-th iterations for $k \geq 0$, $\boldsymbol{\chi}^{(k)} = (\chi_1^{(k)}, \chi_2^{(k)}, \cdots, \chi_n^{(k)})$ and $\boldsymbol{c}^{(k)}$, then find $\boldsymbol{\chi}^{(k+1)} \in \mathcal{S}$ and $\boldsymbol{c}^{(k+1)} \in \mathbb{R}^n$ sequentially such that

$$\begin{aligned} \boldsymbol{\chi}^{(k+1)} &= \underset{\boldsymbol{\chi}\in\mathcal{S}}{\arg\min}\, \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}), \\ \boldsymbol{c}^{(k+1)} &= \underset{\boldsymbol{c}\in\mathbb{R}^n}{\arg\min}\, \mathcal{E}_\tau(\boldsymbol{\chi}^{(k+1)}, \boldsymbol{c}). \end{aligned}$$

The c-subproblem

Note that the energy functional is given by

$$\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}) = \sum_{i=1}^{n} \Big(\lambda \sum_{j=1, j \neq i}^{n} \sqrt{\frac{\pi}{\tau}} \int_\Omega \chi_i(\boldsymbol{x}) G_\tau(\boldsymbol{x}) * \chi_j(\boldsymbol{x}) \, d\boldsymbol{x} + \int_\Omega \chi_i(\boldsymbol{x}) g_i(\boldsymbol{x}) \, d\boldsymbol{x} \Big).$$

Then

$$\min_{\boldsymbol{c} \in \mathbb{R}^n} \mathcal{E}_\tau(\boldsymbol{\chi}^{(k+1)}, \boldsymbol{c}) = \min_{\boldsymbol{c} \in \mathbb{R}^n} \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x}) (f(\boldsymbol{x}) - c_i)^2 \, d\boldsymbol{x}$$

Letting

$$\frac{\partial}{\partial c_i} \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x}) (f(\boldsymbol{x}) - c_i)^2 \, d\boldsymbol{x} = 0,$$

we have

$$-2 \int_\Omega \chi_i^{(k+1)}(\boldsymbol{x}) (f(\boldsymbol{x}) - c_i) \, d\boldsymbol{x} = 0 \Longrightarrow c_i = \frac{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x}) f(\boldsymbol{x}) \, d\boldsymbol{x}}{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x}) \, d\boldsymbol{x}}.$$

The χ-subproblem

Consider the χ-subproblem:

$$\chi^{(k+1)} = \arg\min_{\chi \in \mathcal{S}} \mathcal{E}_\tau(\chi, \boldsymbol{c}^{(k)}).$$

Note that the minimization problem is a non-convex problem since the characteristic function set $\mathcal{S}$ is not a convex set. In order to circumvent this drawback, we define the convex hull $\mathcal{K}$ of $\mathcal{S}$ by

$$\mathcal{K} = \left\{ \chi \in (BV(\Omega))^n : 0 \le \chi_i(\boldsymbol{x}) \le 1, \sum_{i=1}^{n} \chi_i(\boldsymbol{x}) = 1 \ \forall \boldsymbol{x} \in \Omega \setminus \mathcal{C} \right\}.$$

Then we consider the convex relaxed minimization problem instead:

$$\min_{\chi \in \mathcal{K}} \mathcal{E}_\tau(\chi, \boldsymbol{c}^{(k)}).$$

The χ-subproblem (cont'd)

In [WLWW-JCP2017], the authors proved that:

Assume that $\boldsymbol{\chi}^* \in \mathcal{K}$ *is a minimizer of* $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ *on* $\mathcal{K}$*, i.e.,*

$$\mathcal{E}_\tau(\boldsymbol{\chi}^*, \boldsymbol{c}^{(k)}) = \min_{\boldsymbol{\chi} \in \mathcal{K}} \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}).$$

Then $\boldsymbol{\chi}^* \in \mathcal{S}$ *and hence it is also a minimizer of* $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ *on* $\mathcal{S}$*, i.e.,*

$$\mathcal{E}_\tau(\boldsymbol{\chi}^*, \boldsymbol{c}^{(k)}) = \min_{\boldsymbol{\chi} \in \mathcal{S}} \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}).$$

Another approach is to show that $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ is a concave functional on the convex set $\mathcal{K}$. Then minimizers can only be attained at the boundary points of the convex set $\mathcal{K}$, i.e., the subset $\mathcal{S}$.

How to solve the χ-subproblem

Linearizing $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$ at $\boldsymbol{\chi}^{(k)}$, we obtain

$$\begin{aligned} \mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)}) &\approx \mathcal{E}_\tau(\boldsymbol{\chi}^{(k)}, \boldsymbol{c}^{(k)}) + \sum_{i=1}^{n} \int_\Omega \frac{\delta \mathcal{E}_\tau}{\delta \chi_i}\Big|_{\boldsymbol{\chi}=\boldsymbol{\chi}^{(k)}} \Big(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\Big)\, d\boldsymbol{x} \\ &:= \mathcal{E}_\tau(\boldsymbol{\chi}^{(k)}, \boldsymbol{c}^{(k)}) + \sum_{i=1}^{n} \int_\Omega \varphi_i^{(k)}(\boldsymbol{x}) \Big(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\Big)\, d\boldsymbol{x}, \end{aligned}$$

where function $\varphi_i^{(k)}$ is given by

$$0 \le \varphi_i^{(k)}(\boldsymbol{x}) := 2\lambda \sqrt{\frac{\pi}{\tau}} \sum_{j=1, i\neq j}^{n} G_\tau(\boldsymbol{x}) * \chi_j^{(k)}(\boldsymbol{x}) + g_i^{(k)}(\boldsymbol{x}).$$

How to solve the χ-subproblem (cont'd)

Dropping the constant terms in $\mathcal{E}_\tau(\boldsymbol{\chi}, \boldsymbol{c}^{(k)})$, then the $\boldsymbol{\chi}$-subproblem becomes

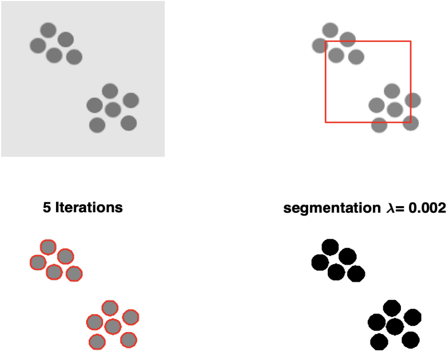
$$\boldsymbol{\chi}^{(k+1)} = \arg\min_{\boldsymbol{\chi}\in\mathcal{K}} \sum_{i=1}^{n} \int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i(\boldsymbol{x})\, d\boldsymbol{x}.$$

Because $\varphi_i^{(k)}(\boldsymbol{x}) \geq 0$ and $\chi_i(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$, the minimizer $\boldsymbol{\chi}^{(k+1)}$ of the above problem can be easily attained at

$$\chi_i^{(k+1)}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \varphi_i^{(k)}(\boldsymbol{x}) = \min\limits_{1\leq \ell \leq n} \varphi_\ell^{(k)}(\boldsymbol{x}), \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{for } i = 1, 2, \cdots, n \text{ and } \boldsymbol{x} \in \Omega \setminus \mathcal{C}.$$

Numerical experiment #1



Numerical experiment #2

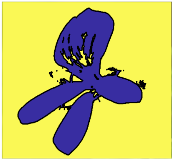
6 Iterations

segmentation λ= 0.005

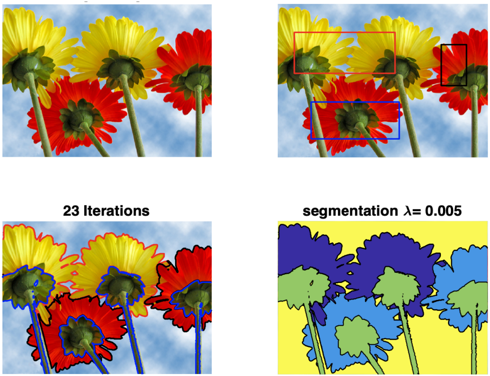
Numerical experiment #3

17 Iterations

segmentation λ= 0.005

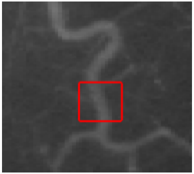


Numerical experiment #4

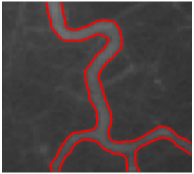


Intensity inhomogeneous images

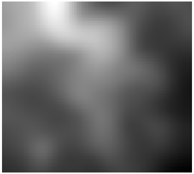
Let $f : \overline{\Omega} \to \mathbb{R}$ be the given grayscale image to be segmented.



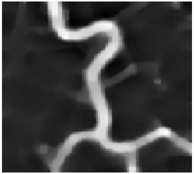
f & initialization C



segmented image



bias field b



corrected image I

Bias field model: $f = bI + n$, where n is the noise

Local intensity clustering model

- C. Li (李純明), R. Huang, Z. Ding, J. C. Gatenby, D. N. Metaxas, and J. C. Gore, A level set method for image segmentation in the presence of intensity inhomogeneities with application to MRI, *IEEE Transactions on Image Processing*, 20 (2011), pp. 2007-2016.
- We need to introduce a bias field model for dealing with intensity inhomogeneous images.
- The level set approach can be replaced by the iterative convolution-thresholding (ICT) scheme.

The bias field model

- The bias field may arise from improper image acquisition in various imaging modalities, especially in the medical imaging domain, such as MRI, PET, CT, etc.
- We assume that the bias field accounts for the intensity inhomogeneity of image. Of course, intensity inhomogeneity can also occur due to spatial variations in illumination.
- The model of bias field in medical images is commonly based upon the assumption that it is *a low-frequency artifact and perceived as a smooth spatially varying function.*
- We assume the multiplicative model with additive noise:

$$f(\boldsymbol{x}) = b(\boldsymbol{x})I(\boldsymbol{x}) + n(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \overline{\Omega},$$

f is the observed image, I the true image, b the bias field, and n an additive zero-mean Gaussian noise, all are unknown except f.

Local intensity clustering property

- Suppose that the true image I approximately takes n distinct constants $c_1, c_2, \cdots, c_n$ in the disjoint regions $\Omega_1, \Omega_2, \cdots, \Omega_n$.
- Let $\boldsymbol{y} \in \Omega$ and

$$\mathcal{N}(\boldsymbol{y}, \rho) := \{\boldsymbol{x} \in \Omega : \ \|\boldsymbol{x} - \boldsymbol{y}\|_2 < \rho\}.$$

  Then $\{\mathcal{N}(\boldsymbol{y}, \rho) \cap \Omega_i\}_{i=1}^n$ forms a natural partition of $\mathcal{N}(\boldsymbol{y}, \rho)$.
- Since b is assumed to be a slowly varying function, it is reasonable that

$$f(\boldsymbol{x}) \approx b(\boldsymbol{y})c_i + n(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathcal{N}(\boldsymbol{y}, \rho) \cap \Omega_i.$$

- The set of these local intensities $\{f(\boldsymbol{x}) |\ \boldsymbol{x} \in \mathcal{N}(\boldsymbol{y}, \rho)\}$ has been naturally *classified into n clusters with the cluster centers $b(\boldsymbol{y})c_i$ in the sense of k-means clustering.*

Local intensity clustering property (cont'd)

- We introduce a nonnegative kernel function $K : \mathbb{R}^2 \to \mathbb{R}$, a truncated Gaussian function,

$$K(\boldsymbol{z}) = \begin{cases} \dfrac{1}{a}\exp\left(-\dfrac{\|\boldsymbol{z}\|_2^2}{2\sigma^2}\right), & \text{for } \|\boldsymbol{z}\|_2 \le \rho, \\ 0, & \text{otherwise,} \end{cases}$$

  $a > 0$ is a normalization constant such that $\int_{\mathbb{R}^2} K(\boldsymbol{z})\, d\boldsymbol{z} = 1$, $\sigma > 0$ is the standard deviation of the Gaussian function.

- We then define a local clustering criterion function $\mathcal{E}(\boldsymbol{y})$ by

$$\mathcal{E}(\boldsymbol{y}) = \sum_{i=1}^{n} \int_{\Omega_i} K(\boldsymbol{y}-\boldsymbol{x})\big(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i\big)^2\, d\boldsymbol{x}.$$

  The smaller the value of $\mathcal{E}(\boldsymbol{y})$, the better the classification of the local intensities $\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{N}(\boldsymbol{y}, \rho)\}$.

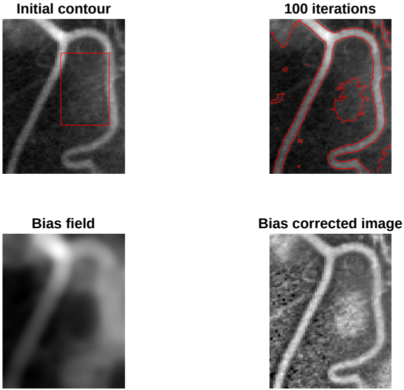
The local intensity clustering model of Li et al.

- Li *et al.* defined the optimal partition $\{\Omega_i\}_{i=1}^{n}$ of Ω as the one such that the local clustering criterion function $\mathcal{E}(\boldsymbol{y})$ is minimized for all $\boldsymbol{y} \in \Omega$.
- They minimized the integral of $\mathcal{E}(\boldsymbol{y})$ with respect to $\boldsymbol{y}$ over Ω, which plays the role of the data fitting term.
- They considered the following local intensity clustering model:

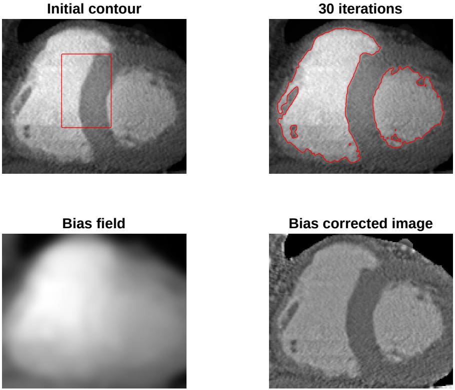
$$\min_{\mathcal{C},b,\boldsymbol{c}} \left(\mu\,|\mathcal{C}| + \int_{\Omega} \sum_{i=1}^{n} \int_{\Omega_i} K(\boldsymbol{y}-\boldsymbol{x}) \big(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i\big)^2 \, d\boldsymbol{x}\, d\boldsymbol{y} \right),$$

  where $\boldsymbol{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$.
- The energy functional is converted to a level set formulation by representing the disjoint regions $\Omega_1, \Omega_2, \cdots, \Omega_n$ with a number of level set functions.

Numerical experiment: level set formulation



Numerical experiment: level set formulation



ICT scheme for solving the model

The ICT scheme can solve the model by considering the following energy functional:

$$\begin{aligned}\mathcal{E}_\tau(\chi, b, \boldsymbol{c}) &= \mu\sqrt{\frac{\pi}{\tau}}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\int_\Omega \chi_i(\boldsymbol{x})\left(G_\tau * \chi_j\right)(\boldsymbol{x})\,d\boldsymbol{x} \\ &\quad + \int_\Omega\sum_{i=1}^{n}\int_\Omega \chi_i(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x})\left(f(\boldsymbol{x}) - b(\boldsymbol{y})c_i\right)^2 d\boldsymbol{x}\,d\boldsymbol{y}.\end{aligned}$$

We consider the minimization problem:

$$\min_{\chi\in\mathcal{S}, b, \boldsymbol{c}} \mathcal{E}_\tau(\chi, b, \boldsymbol{c}).$$

Three subproblems

Divide the minimization problem into three subproblems: find $\chi^{(k+1)} \in \mathcal{S}$, $b^{(k+1)}$, and $\boldsymbol{c}^{(k+1)}$ sequentially such that

$$\begin{aligned}
\chi^{(k+1)} &= \arg\min_{\chi \in \mathcal{S}} \mathcal{E}_1(\chi), \quad \text{where } \mathcal{E}_1(\chi) := \mathcal{E}_\tau(\chi, b^{(k)}, \boldsymbol{c}^{(k)}), \\
b^{(k+1)} &= \arg\min_{b} \mathcal{E}_2(b), \quad \text{where } \mathcal{E}_2(b) := \mathcal{E}_\tau(\chi^{(k+1)}, b, \boldsymbol{c}^{(k)}), \\
\boldsymbol{c}^{(k+1)} &= \arg\min_{\boldsymbol{c}} \mathcal{E}_3(\boldsymbol{c}), \quad \text{where } \mathcal{E}_3(\boldsymbol{c}) := \mathcal{E}_\tau(\chi^{(k+1)}, b^{(k+1)}, \boldsymbol{c}).
\end{aligned}$$

b-subproblem

We set the functional derivative $\delta\mathcal{E}_2/\delta b$ to be zero,

$$\frac{\delta\mathcal{E}_2}{\delta b} = \sum_{i=1}^{n}\int_{\Omega}\chi_i^{(k+1)}(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x})\big(f(\boldsymbol{x})-b(\boldsymbol{y})c_i^{(k)}\big)\big(-2c_i^{(k)}\big)\,d\boldsymbol{x} = 0,$$

which implies

$$b(\boldsymbol{y})\int_{\Omega}\Big(\sum_{i=1}^{n}(c_i^{(k)})^2\chi_i^{(k+1)}(\boldsymbol{x})\Big)K(\boldsymbol{y}-\boldsymbol{x})\,d\boldsymbol{x} = \int_{\Omega}\Big(\sum_{i=1}^{n}c_i^{(k)}\chi_i^{(k+1)}(\boldsymbol{x})\Big)f(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x}$$

Then

$$b^{(k+1)}(\boldsymbol{y}) = \frac{((J_1 f)*K)(\boldsymbol{y})}{(J_2*K)(\boldsymbol{y})} \quad \text{for } \boldsymbol{y}\in\Omega\setminus\mathcal{C},$$

where

$$J_1(\boldsymbol{x}) = \sum_{i=1}^{n}c_i^{(k)}\chi_i^{(k+1)}(\boldsymbol{x}) \quad \text{and} \quad J_2(\boldsymbol{x}) = \sum_{i=1}^{n}(c_i^{(k)})^2\chi_i^{(k+1)}(\boldsymbol{x}).$$

c-subproblem

Setting all the derivatives of $\mathcal{E}_3$ with respect to c_i to be zero, we obtain

$$\begin{aligned}\frac{\partial \mathcal{E}_3}{\partial c_i} &= 2\int_\Omega\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})K(\boldsymbol{y}-\boldsymbol{x})\left(f(\boldsymbol{x})-b^{(k+1)}(\boldsymbol{y})c_i\right)\left(-b^{(k+1)}(\boldsymbol{y})\right)d\boldsymbol{x}d\boldsymbol{y}\\ &= 0.\end{aligned}$$

Since $K(\boldsymbol{y}-\boldsymbol{x}) = K(\boldsymbol{x}-\boldsymbol{y})$, we can exchange the order of integrations,

$$\begin{aligned}&c_i\int_\Omega\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})\left(b^{(k+1)}(\boldsymbol{y})\right)^2 K(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}d\boldsymbol{x}\\ &\qquad = \int_\Omega\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})f(\boldsymbol{x})b^{(k+1)}(\boldsymbol{y})K(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{y}d\boldsymbol{x}.\end{aligned}$$

It leads to

$$c_i^{(k+1)} = \frac{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})f(\boldsymbol{x})\left(b^{(k+1)} * K\right)(\boldsymbol{x})\,d\boldsymbol{x}}{\int_\Omega \chi_i^{(k+1)}(\boldsymbol{x})\left((b^{(k+1)})^2 * K\right)(\boldsymbol{x})\,d\boldsymbol{x}} \quad \text{for } i = 1, 2, \cdots, n.$$

χ-subproblem

Linearizing the energy functional $\mathcal{E}_1(\chi)$ at $\chi^{(k)}$, we have

$$\begin{aligned}\mathcal{E}_1(\chi) &\approx \mathcal{E}_1(\chi^{(k)}) + \sum_{i=1}^{n} \int_\Omega \frac{\delta \mathcal{E}_1}{\delta \chi_i}\Big|_{\chi=\chi^{(k)}} \left(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\right) d\boldsymbol{x} \\ &:= \mathcal{E}_1(\chi^{(k)}) + \sum_{i=1}^{n} \int_\Omega \varphi_i^{(k)}(\boldsymbol{x}) \left(\chi_i(\boldsymbol{x}) - \chi_i^{(k)}(\boldsymbol{x})\right) d\boldsymbol{x}.\end{aligned}$$

where function $\varphi_i^{(k)}$ is given by

$$\begin{aligned}0 \le \varphi_i^{(k)}(\boldsymbol{x}) &:= 2\mu\sqrt{\frac{\pi}{\tau}} \sum_{j=1, j\neq i}^{n} G_\tau(\boldsymbol{x}) * \chi_j^{(k)}(\boldsymbol{x}) \\ &\quad + \int_\Omega K(\boldsymbol{y}-\boldsymbol{x}) \left(f(\boldsymbol{x}) - b^{(k)}(\boldsymbol{y}) c_i^{(k)}\right)^2 d\boldsymbol{y}.\end{aligned}$$

χ-subproblem (cont'd)

We then replace the minimization problem with

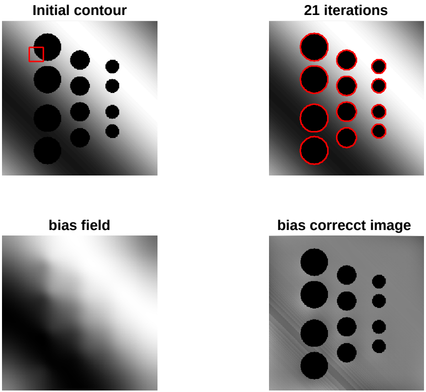
$$\begin{aligned}\chi^{(k+1)} &= \arg\min_{\chi\in\mathcal{K}}\Big(\mathcal{E}_1(\chi^{(k)}) + \sum_{i=1}^{n}\int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i(\boldsymbol{x})\,d\boldsymbol{x} \\ &\qquad - \sum_{i=1}^{n}\int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i^{(k)}(\boldsymbol{x})\,d\boldsymbol{x}\Big) \\ &= \arg\min_{\chi\in\mathcal{K}}\Big(\sum_{i=1}^{n}\int_\Omega \varphi_i^{(k)}(\boldsymbol{x})\chi_i(\boldsymbol{x})\,d\boldsymbol{x}\Big).\end{aligned}$$

Because $\varphi_i^{(k)}(\boldsymbol{x}) \geq 0$ and $\chi_i(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$, the minimizer $\chi^{(k+1)}$ can be easily attained at

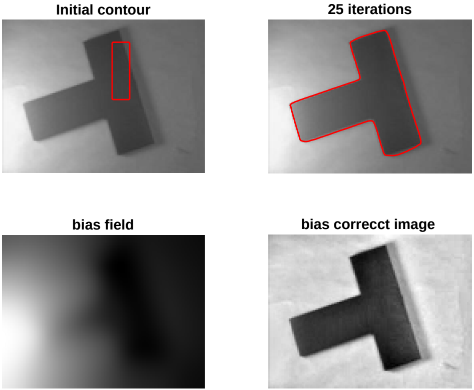
$$\chi_i^{(k+1)}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \varphi_i^{(k)}(\boldsymbol{x}) = \min\limits_{1\leq \ell\leq n} \varphi_\ell^{(k)}(\boldsymbol{x}), \\ 0, & \text{otherwise,}\end{cases}$$

$$\text{for } i = 1, 2, \cdots, n \text{ and } \boldsymbol{x} \in \Omega \setminus \mathcal{C}.$$

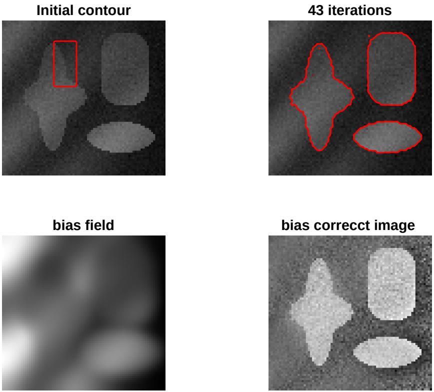
Numerical experiment: ICT scheme



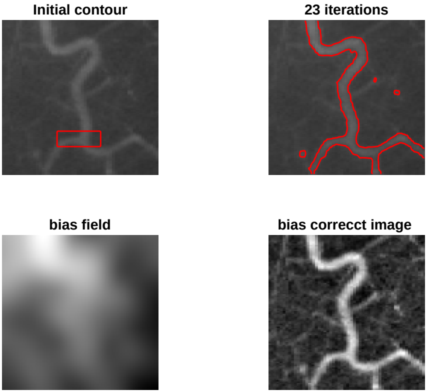
Numerical experiment: ICT scheme



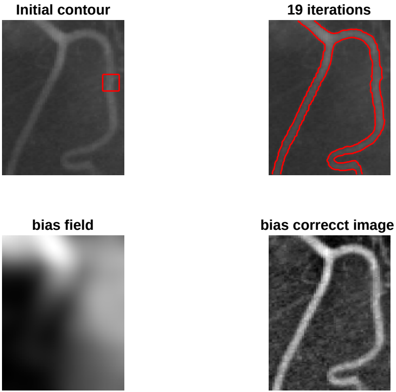
Numerical experiment: ICT scheme



Numerical experiment: ICT scheme



Numerical experiment: ICT scheme



Numerical experiment: ICT scheme