Introduction to Project Topics

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Outline

This course will mainly focus on the following four topics:

1. Numerical methods for PDEs with applications to variational image processing
2. Principal component pursuit problem for low-rank textures
3. Sparse representation and dictionary learning
4. Projection methods for the incompressible Navier-Stokes equations

Topic 1:
Numerical Methods for PDEs with Applications to Variational Image Processing

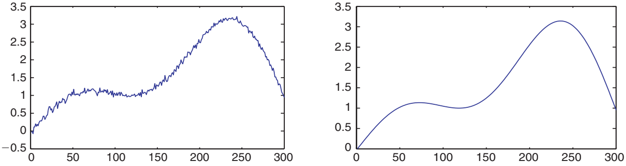
Total variation

Let $u : [a, b] \to \mathbb{R}$. Let $\mathcal{P}_n = \{x_0 = a, x_1, \cdots, x_n = b\}$ be an arbitrary partition of $\overline{\Omega} := [a, b]$ and $\Delta x_i = x_i - x_{i-1}$. The total variation of u is

$$\begin{aligned} \|u\|_{TV(\Omega)} &:= \sup_{\mathcal{P}_n} \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| = \sup_{\mathcal{P}_n} \sum_{i=1}^{n} \left| \frac{u(x_i) - u(x_{i-1})}{\Delta x_i} \right| \Delta x_i \\ &= \int_\Omega |u'(x)|\, dx, \quad \text{if } u \text{ is smooth.} \end{aligned}$$

Denoising is the problem of removing noise from an image:

$$\text{minimize} \left(\int_\Omega |u'(x)|\, dx + \text{ some data fidelity term} \right).$$



Euler-Lagrange equation of the ROF model

Let us consider the following energy minimization problem (Rudin-Osher-Fatemi model):

$$\min_{u \in \mathcal{V}} \left(\int_\Omega |\nabla u| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega \big(u(\boldsymbol{x}) - f(\boldsymbol{x})\big)^2 \, d\boldsymbol{x} \right),$$

where $\mathcal{V}$ is a suitable function space and $\lambda > 0$ is the regularization parameter. Since $\int_\Omega |\nabla u| \, d\boldsymbol{x} = \int_\Omega \sqrt{u_x^2 + u_y^2} \, d\boldsymbol{x}$, we have

$$L(x, y, u, u_x, u_y) = \sqrt{u_x^2 + u_y^2} + \frac{\lambda}{2}(u - f)^2,$$

which leads to the Euler-Lagrange equation with the Neumann boundary condition,

$$-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + \lambda u = \lambda f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega.$$

The Euler-Lagrange equation

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain. We consider the following real-valued energy functional,

$$E[v] := \int_\Omega L(x, y, v(x,y), v_x(x,y), v_y(x,y))\, d\boldsymbol{x},$$

where we assume that $v \in C^2(\overline{\Omega})$ and $L \in C^2$ with respect to its arguments $\boldsymbol{x} = (x, y)$, v, v_x and v_y. According to the fundamental lemma of calculus of variations, we have the following Euler-Lagrange equation,

$$\frac{\partial L}{\partial u} - \nabla \cdot \left(\frac{\partial L}{\partial u_x}, \frac{\partial L}{\partial u_y}\right)^\top = 0 \quad \text{in } \Omega,$$

and the homogeneous Neumann boundary condition,

$$\frac{\partial L}{\partial u_x} n_1 + \frac{\partial L}{\partial u_y} n_2 = 0 \quad \text{on } \partial\Omega.$$

Numerical methods

Therefore, the minimizer of the ROF model can be obtained by

- **Nonlinear PDE-based method:** evolving a finite difference approximation of the parabolic partial differential equation with the homogeneous Neumann BC to reach a steady state solution:

$$\overbrace{\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda u = \lambda f}^{\text{Heat-type equation}} \quad \text{for } (t, \boldsymbol{x}) \in (0, T) \times \Omega,$$

$$u(0, \boldsymbol{x}) = f(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \overline{\Omega},$$

$$\nabla u \cdot \boldsymbol{n} = 0 \quad \text{for } t \in [0, T] \text{ and } \boldsymbol{x} \in \partial\Omega.$$

- **An alternating direction approach – split Bregman method:** Introducing the new unknown vector function d, we have the constrained minimization problem:

$$\min_{u,d} \left(\int_\Omega |d| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega (u(\boldsymbol{x}) - f(\boldsymbol{x}))^2 \, d\boldsymbol{x} \right) \quad \text{subject to } d = \nabla u.$$

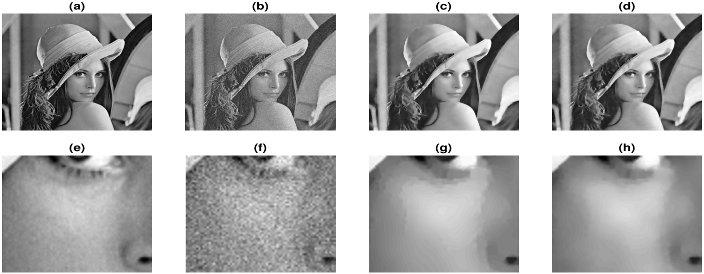
ROF total-variation model vs. adaptive diffusivity model

Let $f : \overline{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ be a given noisy image. Rudin-Osher-Fatemi (1992) proposed the model:

$$\min_{u \in \mathcal{V}} \left(\int_\Omega |\nabla u| + \frac{\lambda}{2} (u - f)^2 \, d\boldsymbol{x} \right), \quad \lambda > 0.$$

Hsieh-Shao-Yang (2018) proposed an adaptive model to alleviate *the staircasing effect:*

$$\min_{u \in \mathcal{V}} \left(\int_\Omega \frac{1}{2} \varphi(|\nabla u^*|) |\nabla u|^2 + \frac{\lambda}{2} (u - f)^2 \, d\boldsymbol{x} \right), \quad \lambda > 0.$$



A variational model for image contrast enhancement

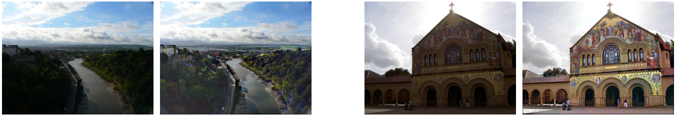
Hsieh-Shao-Yang (2020): for every $f \in \{f_R, f_G, f_B\}$, we solve

$$\min_{u \in \mathcal{V}} \left(\int_\Omega |\nabla u - \nabla h_c| \, d\boldsymbol{x} + \frac{\lambda}{2} \int_\Omega (u - g_c)^2 \, d\boldsymbol{x} \right),$$

where the adaptive functions g_c and h_c are defined as

$$g_c(\boldsymbol{x}) := \begin{cases} \alpha \bar{f}, & \boldsymbol{x} \in \Omega_d, \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b, \end{cases} \qquad h_c(\boldsymbol{x}) := \begin{cases} \beta f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_d, \\ f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_b. \end{cases}$$

Numerical methods: *(i) Euler-Lagrange equation + solving IBVP; (ii) direct discretization + split Bregman iterations.*



Numerical results by the split Bregman iterations

Mumford-Shah image segmentation model

Mumford-Shah model: it finds a piecewise smooth function u and a curve set $\mathcal{C}$, which separates the image domain into disjoint regions, minimizing the energy functional:

$$\min_{u,\mathcal{C}} \left(\mu|\mathcal{C}| + \lambda \int_{\Omega} (f(\mathbf{x}) - u(\mathbf{x}))^2 \, d\mathbf{x} + \int_{\Omega \setminus \mathcal{C}} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \right),$$

where $|\mathcal{C}|$ denotes the total length of the curves in $\mathcal{C}$.

- The first term plays the regularization role, which ensures the target objects can tightly be wrapped by $\mathcal{C}$.
- The second term is the data fidelity term, which forces u to be close to the input image f.
- The third term is the smoothing term, which forces the target function u to be piecewise smooth within each of the regions separated by the curves in $\mathcal{C}$.
- $\mu > 0$, $\lambda > 0$ *are tuning parameters to modulate these three terms.*

Chan-Vese two-phase model

In 1999, Chan and Vese proposed a two-phase segmentation model based on the level set formulation:

$$\min_{c_1,c_2,\mathcal{C}} \left(\mu|\mathcal{C}| + \nu|\Omega_{\text{in}}| + \lambda_1 \int_{\Omega_{\text{in}}} \left(f(\boldsymbol{x}) - c_1\right)^2 d\boldsymbol{x} + \lambda_2 \int_{\Omega_{\text{out}}} \left(f(\boldsymbol{x}) - c_2\right)^2 d\boldsymbol{x}\right).$$

- Ω_{in} denotes the region enclosed by the curves in $\mathcal{C}$ with area $|\Omega_{\text{in}}|$, and $\Omega_{\text{out}} := \Omega \setminus \Omega_{\text{in}}$.
- $\mu > 0$, $\nu \geq 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ are tuning parameters (actually, one of them can be fixed as 1).
- Chan-Vese model finds a piecewise constant function u and a curve set $\mathcal{C}$ to minimize the energy functional, where u has only two constant values,

$$u(\boldsymbol{x}) = \begin{cases} c_1, & \boldsymbol{x} \text{ is inside } \mathcal{C}, \\ c_2, & \boldsymbol{x} \text{ is outside } \mathcal{C}. \end{cases}$$

Level set function

Therefore, we represent $\mathcal{C}$ implicitly by the zero level contour of a level set function $\phi : \overline{\Omega} \to \mathbb{R}$, i.e.,

$$\mathcal{C} = \{\boldsymbol{x} \in \overline{\Omega} : \ \phi(\boldsymbol{x}) = 0\}.$$

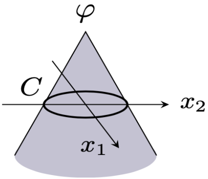
The zero level contour $\mathcal{C}$ partitions the image domain into two disjoint regions Ω_{in} and Ω_{out} such that

$$\phi(\boldsymbol{x}) > 0 \ \text{ for } \ \boldsymbol{x} \in \Omega_{\text{in}} \quad \text{and} \quad \phi(\boldsymbol{x}) < 0 \ \text{ for } \ \boldsymbol{x} \in \Omega_{\text{out}}.$$

For example, given $r > 0$, we define a level set function

$$\phi(\boldsymbol{x}) = \phi(x, y) = r - \sqrt{x^2 + y^2},$$

whose zero level contour is the circle of radius $r > 0$.



Chan-Vese two-phase model

Let H denote the Heaviside function and δ the Dirac delta function,

$$H(s) = \begin{cases} 1 & s \geq 0, \\ 0 & s < 0, \end{cases} \quad \text{and} \quad \frac{d}{ds}H(s) = \delta(s).$$

Then the Chan-Vese two-phase model has the form

$$\min_{c_1,c_2,\phi} \Bigg(\underbrace{\mu \int_\Omega \delta(\phi(\boldsymbol{x}))|\nabla\phi(\boldsymbol{x})|\,d\boldsymbol{x}}_{=\mu\int_\Omega |\nabla H(\phi(\boldsymbol{x}))|\,d\boldsymbol{x} = \mu|\mathcal{C}|} + \underbrace{\nu \int_\Omega H(\phi(\boldsymbol{x}))\,d\boldsymbol{x}}_{=\nu|\Omega_{\text{in}}|}$$

$$+ \underbrace{\lambda_1 \int_\Omega \big(f(\boldsymbol{x}) - c_1\big)^2 H(\phi(\boldsymbol{x}))\,d\boldsymbol{x}}_{=\lambda_1 \int_{\Omega_{\text{in}}} (f(\boldsymbol{x})-c_1)^2\,d\boldsymbol{x}}$$

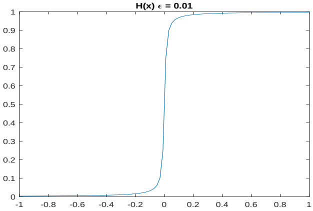
$$+ \underbrace{\lambda_2 \int_\Omega \big(f(\boldsymbol{x}) - c_2\big)^2 \big(1 - H(\phi(\boldsymbol{x}))\big)\,d\boldsymbol{x}}_{=\lambda_2 \int_{\Omega_{\text{out}}} (f(\boldsymbol{x})-c_2)^2\,d\boldsymbol{x}} \Bigg).$$

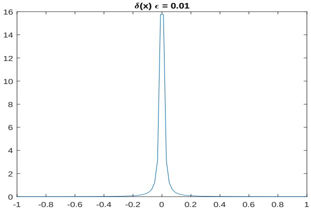
Regularized Heaviside and delta functions

The Heaviside function H and the Dirac delta function δ can be approximately regularized as follows: for a sufficiently small $\epsilon > 0$,

$$H_\epsilon(t) := \frac{1}{2}\left(1 + \frac{2}{\pi}\tan^{-1}\left(\frac{t}{\epsilon}\right)\right), \quad \delta_\epsilon(t) := \frac{d}{dt}H_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2 + t^2)},$$

$$\int_{-\infty}^{\infty} \delta_\epsilon(t)dt = \int_{-\infty}^{\infty} \frac{\epsilon}{\pi(\epsilon^2 + t^2)}dt = \cdots = 1.$$





An alternating iterative scheme

The minimization is solved by *an alternating iterative scheme*, i.e., alternatingly updating c_1, c_2 and ϕ.

(S1) Fixed ϕ, the optimal values of c_1 and c_2 are the region averages,

$$c_1 = \frac{\int_\Omega f(\boldsymbol{x}) H(\phi(\boldsymbol{x}))\, d\boldsymbol{x}}{\int_\Omega H(\phi(\boldsymbol{x}))\, d\boldsymbol{x}}, \quad c_2 = \frac{\int_\Omega f(\boldsymbol{x}) \big(1 - H(\phi(\boldsymbol{x}))\big)\, d\boldsymbol{x}}{\int_\Omega \big(1 - H(\phi(\boldsymbol{x}))\big)\, d\boldsymbol{x}}.$$

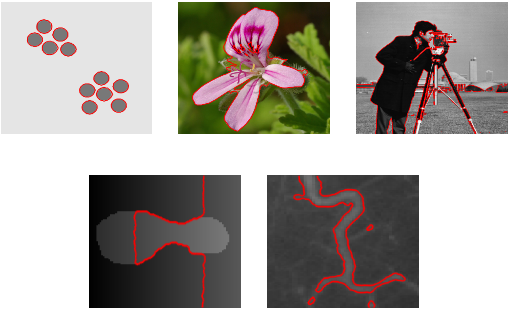
(S2) Fixed c_1, c_2, we solve the initial-boundary value problem (IBVP) for the Euler-Lagrange equation to reach a steady-state solution:

$$\frac{\partial \phi}{\partial t} = \delta_\epsilon(\phi)\left(\mu \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} - \nu - \lambda_1 (f - c_1)^2 + \lambda_2 (f - c_2)^2\right),$$
$$\text{for } \ t > 0, \boldsymbol{x} \in \Omega,$$

$$\phi(0, \boldsymbol{x}) = \phi_0(\boldsymbol{x}), \boldsymbol{x} \in \Omega,$$

$$\frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega, t \geq 0.$$

Numerical experiments of the Chan-Vese model



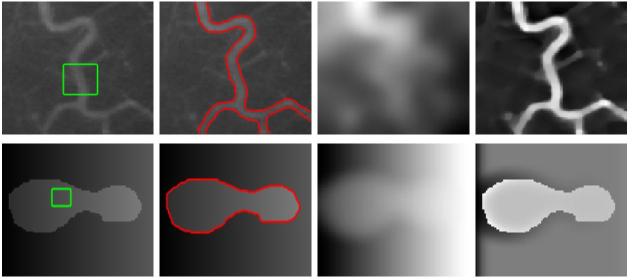
numerical results by the alternating iterative scheme

Adaptive model for intensity inhomogeneous images

Liao-Yang-You (2022) proposed an entropy-weighted local intensity clustering-based model starting from *the bias field model: $f = bI + n$:*

$$\min_{\mathcal{C},b,\mathbf{c}} \left(\mu \left|\mathcal{C}\right| + \int_{\Omega} E_r(\mathbf{y}) \sum_{i=1}^{n} \int_{\Omega_i} K(\mathbf{y}-\mathbf{x}) \big(f(\mathbf{x}) - b(\mathbf{y})c_i\big)^2 \, d\mathbf{x}\, d\mathbf{y} \right).$$

Numerical method: *a new alternating iterative scheme, called iterative convolution-thresholding (ICT) scheme.*



initial contour, segmented result, bias field b, and corrected image f/b

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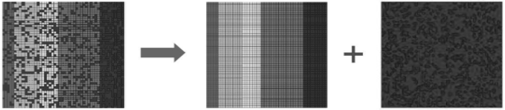
Topic 2:
Principal Component Pursuit Problem for Low-Rank Textures

Sparse plus low rank matrix decomposition

Let $\boldsymbol{M} \in \mathbb{R}^{m\times n}$ be a given grayscale image. Suppose that $\boldsymbol{M}$ is the superposition of a low-rank component $\boldsymbol{L}$ and a sparse component $\boldsymbol{S}$,

$$\boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S}.$$

We are interested in finding the low-rank image $\boldsymbol{L}$, which has high repeatability along horizontal or vertical directions.



The sparse plus low rank decomposition problem can be formulated as the constrained minimization problem:

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\operatorname{rank}(\boldsymbol{L}) + \lambda \|\boldsymbol{S}\|_0\right) \quad \text{subject to} \quad \boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S},$$

where $\lambda > 0$ is a tuning parameter and $\|\boldsymbol{S}\|_0$ denotes the number of non-zero entries in $\boldsymbol{S}$. *The problem is not convex.*

The principal component pursuit problem

We approximate the sparse plus low rank decomposition problem by the following *principal component pursuit (PCP) problem:*

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \right) \quad \text{subject to} \quad \boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S},$$

where $\|\boldsymbol{L}\|_*$ is the nuclear (Ky Fan/樊(土畿)) norm of $\boldsymbol{L}$ defined as

$$\|\boldsymbol{L}\|_* := \sum_{i=1}^{r} \sigma_i,$$

and $r \in \mathbb{N}^+$ is the rank of $\boldsymbol{L}$ and σ_i are the singular values of $\boldsymbol{L}$, and $\|\boldsymbol{S}\|_1$ denotes the ℓ^1-norm of $\boldsymbol{S}$ *(seen as a long vector in* $\mathbb{R}^{mn}$*),*

$$\|\boldsymbol{S}\|_1 := \sum_{i,j} |S_{ij}|.$$

The penalty formulation and alternating direction method

Let $\mu > 0$ be the penalty parameter. Then we consider the relaxation using a penalty term to replace the constraint,

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L} - \boldsymbol{S}\|_F^2 \right),$$

where $\|\cdot\|_F$ is the Frobenius norm. We set, for example, $\boldsymbol{S}^{(0)} = \mathbf{0}$. The ADM for the penalty formulation is given as follows: for $k \geq 0$, find

$$\begin{aligned}
\boldsymbol{L}^{(k+1)} &= \arg\min_{\boldsymbol{L}} \left(\|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}^{(k)}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L} - \boldsymbol{S}^{(k)}\|_F^2 \right), \\
\boldsymbol{S}^{(k+1)} &= \arg\min_{\boldsymbol{S}} \left(\|\boldsymbol{L}^{(k+1)}\|_* + \lambda \|\boldsymbol{S}\|_1 + \frac{\mu}{2} \|\boldsymbol{M} - \boldsymbol{L}^{(k+1)} - \boldsymbol{S}\|_F^2 \right).
\end{aligned}$$

By further analysis, we can prove that

$$\begin{aligned}
\boldsymbol{L}^{(k+1)} &= \mathrm{SVT}_{\frac{1}{\mu}} \left(\boldsymbol{M} - \boldsymbol{S}^{(k)} \right), \\
\boldsymbol{S}^{(k+1)} &= \mathrm{sign}(\boldsymbol{M} - \boldsymbol{L}^{(k+1)}) \odot \max\left\{ |\boldsymbol{M} - \boldsymbol{L}^{(k+1)}| - (\lambda/\mu),\ 0 \right\},
\end{aligned}$$

where $\odot$ is the Hadamard product (i.e., element-wise product).

SVD and SVT

- **Singular value decomposition (SVD)**

  Let $\boldsymbol{M} \in \mathbb{R}^{m\times n}$. The SVD of $\boldsymbol{M}$ is the factorization in the form

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top,$$

  where $\boldsymbol{U} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ are orthogonal matrices ($\boldsymbol{U}\boldsymbol{U}^\top = \boldsymbol{I}$ and $\boldsymbol{V}\boldsymbol{V}^\top = \boldsymbol{I}$) and $\boldsymbol{\Sigma} \in \mathbb{R}^{m\times n}$ is diagonal with all non-negative entries called the singular values of $\boldsymbol{M}$.

- **Singular value thresholding (SVT)**

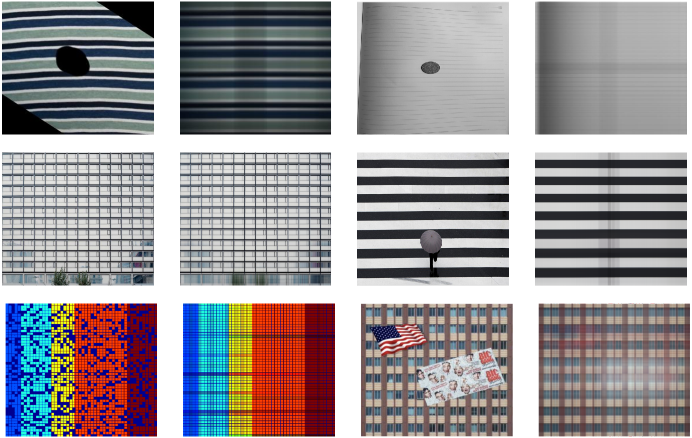
  Let $\boldsymbol{M} \in \mathbb{R}^{m\times n}$. Suppose that the SVD of $\boldsymbol{M}$ is given by $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$. Then the singular value thresholding (SVT) of $\boldsymbol{M}$ with threshold $\tau > 0$ is defined by

$$SVT_\tau(\boldsymbol{M}) = \boldsymbol{U}\boldsymbol{D}_\tau(\boldsymbol{\Sigma})\boldsymbol{V}^\top,$$

  where

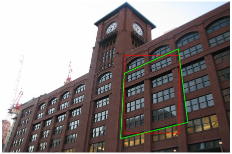
$$\boldsymbol{D}_\tau(\boldsymbol{\Sigma})_{ii} = \max\{\boldsymbol{\Sigma}_{ii} - \tau,\ 0\}.$$

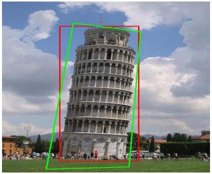
Background recovering using the penalty method



Some project topics for PCP problem

- Implement the principal component pursuit problem for low-rank textures by the penalty method, the augmented Lagrange multiplier method, etc.
- Further study of the transform invariant low-rank textures:

 $\Rightarrow$ 

 $\Rightarrow$ 

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Topic 3:
Sparse Representation and Dictionary Learning

Sparse representation problem

Terms: Sparse Representation (稀疏表現)/Sparse Coding (稀疏編碼)

SR problem: *Given a signal vector* $\boldsymbol{x} \in \mathbb{R}^m$ *and a dictionary matrix* $\boldsymbol{D} \in \mathbb{R}^{m \times n}$*, we seek a sparse coefficient vector* $\boldsymbol{z}^* \in \mathbb{R}^n$ *such that*

$$\boldsymbol{z}^* = \arg\min_{\boldsymbol{z}} \left(\frac{1}{2} \|\boldsymbol{x} - \boldsymbol{D}\boldsymbol{z}\|_2^2 + \lambda \|\boldsymbol{z}\|_0 \right),$$

where $\lambda > 0$ *is a penalty parameter and* $\|\boldsymbol{z}\|_0$ *counts the number of nonzero components of* $\boldsymbol{z}$*.*

Remarks:

- In the matrix-vector multiplication $\boldsymbol{D}\boldsymbol{z}$, the components of $\boldsymbol{z}$ are the coefficients with respect to columns (also called *atoms*) of $\boldsymbol{D}$.
- We call $\|\boldsymbol{z}\|_0$ the ℓ^0 norm of $\boldsymbol{z}$, even though ℓ^0 is *not* really a norm, since the *homogeneity property* fails, $\|\alpha \boldsymbol{z}\|_0 \neq |\alpha| \|\boldsymbol{z}\|_0$.
- It is inefficient to compute $\|\boldsymbol{z}\|_0$ directly when n is large. In practice, we will use the ℓ^1 norm instead of the ℓ^0 norm.

The ℓ^1-norm SR problem

- ℓ^1**-norm SR problem:** *Given a signal vector* $\boldsymbol{x} \in \mathbb{R}^m$ *and a dictionary matrix* $\boldsymbol{D} \in \mathbb{R}^{m \times n}$*, we seek a coefficient vector* $\boldsymbol{z}^* \in \mathbb{R}^n$ *such that*

$$\boldsymbol{z}^* = \arg\min_{\boldsymbol{z} \in \mathbb{R}^n} \left(\frac{1}{2} \|\boldsymbol{x} - \boldsymbol{D}\boldsymbol{z}\|_2^2 + \lambda \|\boldsymbol{z}\|_1 \right), \qquad \lambda > 0. \qquad (\star)$$

  The existence (and uniqueness) of solution of the problem $(\star)$ can be ensured because matrix $\boldsymbol{D}^\top \boldsymbol{D}$ is symmetric (+ *positive definite*) and the second term $\lambda \|\cdot\|_1$ is a *convex function.*

- Problem $(\star)$ is also a regression analysis method in statistics and machine learning. It is the so-called *least absolute shrinkage and selection operator (LASSO).*

  R. J. Tibshirani, The lasso problem and uniqueness, *Electronic Journal of Statistics,* 7 (2013), pp. 1456-1490 ⊕ A. Ali, 13 (2019), pp. 2307-2347.

Alternating direction method of multipliers (ADMM)

- For the ℓ^1-norm SR problem,

$$z^* = \arg\min_{z}\left(\frac{1}{2}\|x - Dz\|_2^2 + \lambda\|z\|_1\right), \qquad \lambda > 0, \qquad (\star)$$

we set

$$f(z) := \frac{1}{2}\|x - Dz\|_2^2,\ g(y) := \lambda\|y\|_1,\ Az + By = c \Leftrightarrow z - y = \mathbf{0}.$$

- The ADMM for the ℓ^1-norm SR problem is given by

$$z^{(i+1)} = \arg\min_{z}\left(\frac{1}{2}\|x - Dz\|_2^2 + \frac{\rho}{2}\|z - y^{(i)} + u^{(i)}\|_2^2\right), \qquad (A1)$$

$$y^{(i+1)} = \arg\min_{y}\left(\lambda\|y\|_1 + \frac{\rho}{2}\|z^{(i+1)} - y + u^{(i)}\|_2^2\right), \qquad (A2)$$

$$u^{(i+1)} = u^{(i)} + \rho(z^{(i+1)} - y^{(i+1)}), \qquad (A3)$$

where $\rho > 0$ is the another *penalty parameter.*

Solving minimization problem $(A1)$

Define

$$F_1(\boldsymbol{z}) := \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{D}\boldsymbol{z}\|_2^2 + \frac{\rho}{2}\|\boldsymbol{z} - \boldsymbol{y}^{(i)} + \boldsymbol{u}^{(i)}\|_2^2.$$

Then F_1 is a quadratic function in variables $z_1, z_2, \cdots, z_n$ and $F_1(\boldsymbol{z}) \geq 0 \;\forall\, \boldsymbol{z} \in \mathbb{R}^n$. To solve "$\min\limits_{\boldsymbol{z}} F_1(\boldsymbol{z})$", first we compute

$$\begin{aligned} \nabla F_1(\boldsymbol{z}) &= -\boldsymbol{D}^\top(\boldsymbol{x} - \boldsymbol{D}\boldsymbol{z}) + \rho \boldsymbol{I}(\boldsymbol{z} - \boldsymbol{y}^{(i)} + \boldsymbol{u}^{(i)}) \\ &= (\boldsymbol{D}^\top \boldsymbol{D} + \rho \boldsymbol{I})\boldsymbol{z} - \left(\boldsymbol{D}^\top \boldsymbol{x} + \rho(\boldsymbol{y}^{(i)} - \boldsymbol{u}^{(i)})\right). \end{aligned}$$

Letting $\nabla F_1(\boldsymbol{z}) = \boldsymbol{0}$, we have

$$(\boldsymbol{D}^\top \boldsymbol{D} + \rho \boldsymbol{I})\boldsymbol{z} = \left(\boldsymbol{D}^\top \boldsymbol{x} + \rho(\boldsymbol{y}^{(i)} - \boldsymbol{u}^{(i)})\right).$$

Therefore, we obtain the solution

$$\boldsymbol{z}^{(i+1)} = (\boldsymbol{D}^\top \boldsymbol{D} + \rho \boldsymbol{I})^{-1}\left(\boldsymbol{D}^\top \boldsymbol{x} + \rho(\boldsymbol{y}^{(i)} - \boldsymbol{u}^{(i)})\right).$$

Solving minimization problem $(A2)$

Using the *soft-thresholding function* $\mathcal{S}_{\lambda/\rho}$, problem $(A2)$ has the closed form solution:

$$\boldsymbol{y}^{(i+1)} = \mathcal{S}_{\lambda/\rho}(\boldsymbol{z}^{(i+1)} + \boldsymbol{u}^{(i)}),$$

where

$$\mathcal{S}_{\lambda/\rho}(\boldsymbol{v}) = \mathrm{sign}(\boldsymbol{v}) \odot \max(\boldsymbol{0},\ |\boldsymbol{v}| - \lambda/\rho),$$

and $\mathrm{sign}(\cdot)$, $\max(\cdot,\cdot)$, and $|\cdot|$ are all applied to the input vector $\boldsymbol{v}$ component-wisely, and $\odot$ is the Hadamard product.

Finally, the iterative scheme can be posed as follows:

$$\begin{aligned}
\boldsymbol{z}^{(i+1)} &= (\boldsymbol{D}^\top \boldsymbol{D} + \rho \boldsymbol{I})^{-1} (\boldsymbol{D}^\top \boldsymbol{x} + \rho(\boldsymbol{y}^{(i)} - \boldsymbol{u}^{(i)})),\\
\boldsymbol{y}^{(i+1)} &= \mathcal{S}_{\lambda/\rho}(\boldsymbol{z}^{(i+1)} + \boldsymbol{u}^{(i)}),\\
\boldsymbol{u}^{(i+1)} &= \boldsymbol{u}^{(i)} + \rho(\boldsymbol{z}^{(i+1)} - \boldsymbol{y}^{(i+1)}).
\end{aligned}$$

Sparse dictionary learning

SDL problem: *Let* $\{\boldsymbol{x}_i\}_{i=1}^{N} \subset \mathbb{R}^m$ *be a given dataset of signals. We seek a dictionary matrix* $\boldsymbol{D} = [\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_n] \in \mathbb{R}^{m\times n}$ *together with the sparse coefficient vectors* $\{\boldsymbol{z}_i\}_{i=1}^{N} \subset \mathbb{R}^n$ *that solve the minimization problem:*

$$\min_{\boldsymbol{D},\{\boldsymbol{z}_i\}} \left(\frac{1}{2} \sum_{i=1}^{N} \|\boldsymbol{x}_i - \boldsymbol{D}\boldsymbol{z}_i\|_2^2 + \lambda \sum_{i=1}^{N} \|\boldsymbol{z}_i\|_1 \right)$$

$$\text{subject to } \|\boldsymbol{d}_k\|_2 \le 1,\ \forall\, 1 \le k \le n, \qquad \lambda > 0.$$

Numerical method: *alternating direction method of multipliers (ADMM).*

Some project topics for SR and DL

Single image inpainting: *we use the complete patches to train the dictionary, recover the incomplete patches by the sparse representation.*

Other applications: *single image super-resolution, image fusion, ...*

References and source codes

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2. M. Elad, *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing,* Springer, New York, 2010.
3. Y. Sharon, J. Wright, and Y. Ma, Computation and relaxation of conditions for equivalence between ℓ^1 and ℓ^0 minimization, *UIUC Technical Report UILU-ENG-07-2008, 2007.*
4. **Sparse dictionary learning:**

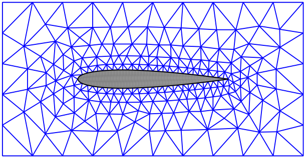
   https://en.wikipedia.org/wiki/Sparse_dictionary_learning
5. **Matlab codes:**

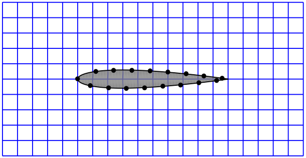
   http://brendt.wohlberg.net/software/SPORCO/

Topic 4:
Projection Methods for the Incompressible Navier-Stokes Equations

Fluid-structure interaction problem (流構耦合問題)

- For computational fluid dynamics (CFD), the primary issues are accuracy, computational efficiency, and the ability to handle complex geometries.
- A fluid-structure interaction (FSI) problem describes the coupled dynamics of fluid mechanics and structure mechanics.
- It usually requires the modeling of complex geometric structure and moving boundaries. It is very challenging for conventional body-fitted approach.





- *We will introduce a Cartesian grid based non-boundary conforming approach, the direct-forcing immersed boundary projection methods.*

Time-dependent incompressible Navier-Stokes equations

Let Ω be an open bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, and let $[0, T]$ be the time interval. The time-dependent, incompressible Navier-Stokes problem can be posed as: find $\boldsymbol{u}$ and p with $\int_\Omega p = 0$, so that

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \times (0, T], \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t = 0\}. \end{aligned}$$

- $\boldsymbol{u}$ is the velocity field, p the pressure (divided by a constant density ρ), ν the kinematic viscosity, $\boldsymbol{f}$ the density of body force.
- *By the divergence theorem, boundary velocity $\boldsymbol{u}_b$ must satisfy*

$$\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} \, dA = \int_\Omega \nabla \cdot \boldsymbol{u} \, dV = 0, \quad \forall \, t \in [0, T].$$

Time-discretization of the incompressible NS equations

First, we discretize the time variable of the Navier-Stokes problem, with the spatial variable being left continuous. Consider the implicit Euler time-discretization with explicit first-order approximation to the nonlinear convection term:

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $t_i := i\Delta t$ for $i = 0, 1, \cdots$, $\Delta t > 0$ is the time step length, and $\boldsymbol{g}^n$ denotes an approximate (or exact) value of $\boldsymbol{g}(t_n)$ at the time level n.

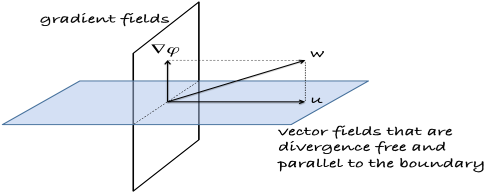
It is highly inefficient in solving this coupled system of Stokes-like equations directly. This is precisely the reason for proposing the projection approach to decouple the computation of $(\boldsymbol{u}^{n+1}, p^{n+1})$.

Helmholtz-Hodge decomposition

Let Ω be an open, bounded, connected, Lipschitz-continuous domain. A vector field $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$ can be uniquely decomposed orthogonally as

$$\boldsymbol{w} = \boldsymbol{u} + \nabla\varphi, \quad \boldsymbol{u} \in H(\text{div};\Omega) \text{ and } \varphi \in H^1(\Omega),$$

where $\boldsymbol{u}$ has zero divergence $\nabla \cdot \boldsymbol{u} = 0$ in Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.



- Orthogonality: $\int_\Omega \boldsymbol{u} \cdot \nabla\varphi \, dV = 0$ (L^2-inner product)
- The HHD describes the decomposition of a flow field $\boldsymbol{w}$ into its divergence-free component $\boldsymbol{u}$ and curl-free component $\nabla\varphi$.
- A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics, 2nd Edition*, Springer-Verlag, New York, 1990.

Chorin projection scheme (Math. Comp. 1968/69)

Step 1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega, \end{cases}$$

which is equivalent to solving the pressure-Poisson equation with the homogeneous Neumann boundary condition:

$$\begin{cases} \nabla^2 p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \boldsymbol{u}^* & \text{in } \Omega, \\ \nabla p^{n+1} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and then define the velocity field by $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1}$.

Remarks on Chorin's first-order scheme

- *The second step is usually referred to as the projection step.*

$$\boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \Delta t \nabla p^{n+1} = \boldsymbol{u}^{n+1} + \nabla(\Delta t p^{n+1}).$$

  This is indeed the standard HHD of $\boldsymbol{u}^*$ when $\boldsymbol{u}_b^{n+1} = \mathbf{0}$ on $\partial\Omega$.

- Summing all equations in Chorin's projection scheme, we have

$$\begin{aligned}
\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} \cdot \boldsymbol{n} &= \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega,
\end{aligned}$$

  different from the original semi-implicit discretization. Since

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1} \approx \boldsymbol{u}^* \textit{ in } \Omega \quad \textit{as } \Delta t \to 0^+,$$

  it is not surprising that we should expect

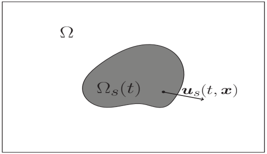
$$\nabla^2 \boldsymbol{u}^{n+1} \approx \nabla^2 \boldsymbol{u}^* \textit{ in } \Omega \quad \textit{and} \quad \boldsymbol{u}^{n+1} \approx \boldsymbol{u}_b^{n+1} \textit{ on } \partial\Omega \quad \textit{as } \Delta t \to 0^+.$$

Fluid-solid interaction (FSI) problem

A simple one-way coupling FSI problem is flow over a stationary or moving solid body with a prescribed velocity.

Let Ω be the fluid domain which encloses a rigid body positioned at $\overline{\Omega}_s(t)$ *with a prescribed velocity* $\boldsymbol{u}_s(t, \boldsymbol{x})$. The FSI problem with initial value and no-slip boundary condition can be posed as follows:

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times (0, T], \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times (0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_s \quad \text{on } \partial\Omega_s \times [0, T], \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times \{t = 0\}, \end{aligned}$$



where $\boldsymbol{u}$ is the velocity field, p the pressure (divided by a constant density ρ), ν the kinematic viscosity, $\boldsymbol{f}$ the density of body force.

The body-fitted approach

The body-fitted approach is a conventional method for solving the FSI problem. For example, using the semi-implicit discretization at time $t = t_{n+1}$, we solve in the fluid domain $\Omega \setminus \overline{\Omega}_s^{n+1}$ the system

$$\begin{aligned}
\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega \setminus \overline{\Omega}_s^{n+1}, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega \setminus \overline{\Omega}_s^{n+1}, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_s^{n+1} \quad \text{on } \partial\Omega_s^{n+1},
\end{aligned}$$

where $t_i := i\Delta t$ for $i = 0, 1, \cdots$, $\Delta t > 0$ is the time step length, and $\boldsymbol{g}^n$ denotes an approximate or exact value of $\boldsymbol{g}(t_n)$ at the time level n.

It is highly inefficient in solving these equations directly when the solid body $\overline{\Omega}_s$ has a complex geometry or moves in the fluid. Below, we will consider a direct-forcing immersed boundary (IB) projection approach.

Direct-forcing immersed boundary (IB) approach

We first consider the solid object as a portion of the fluid and then introduce a virtual force $\boldsymbol{F}$ to the momentum equation, and we expect the problem can be solved on the whole domain Ω and do not need to set the interior boundary condition $\boldsymbol{u} = \boldsymbol{u}_s$ on the interface $\partial\Omega_s$:

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} + \boldsymbol{F} \quad \text{in } \Omega \times (0, T], \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t = 0\}.
\end{aligned}
$$

- *Note that the virtual force $\boldsymbol{F}$ is distributed only in the whole solid object region $\overline{\Omega}_s(t)$, making the region acts exactly as if it were a solid rigid body immersed in the fluid with a prescribed velocity $\boldsymbol{u}_s(t, \boldsymbol{x})$.*
- *But, at this moment, we do not know how to specify the virtual force $\boldsymbol{F}$ such that the region fulfills the prescribed velocity $\boldsymbol{u}_s(t, \boldsymbol{x})$.*

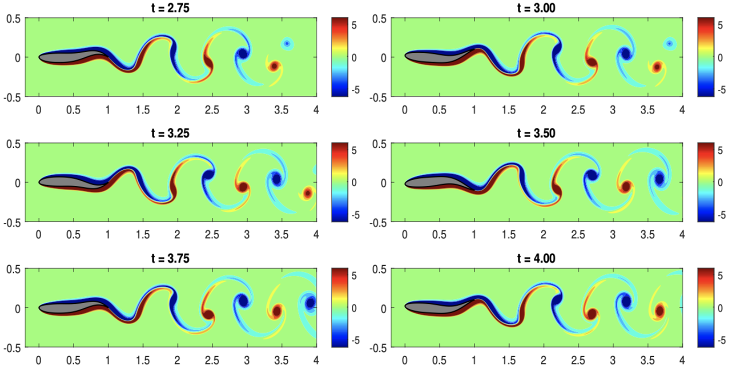
Time-discretization of the incompressible N-S equations

Let us first discretize the time variable of the Navier-Stokes problem, with the spatial variable being left continuous. Consider the implicit Euler time-discretization with an explicit first-order approximation to the nonlinear convection. Then we have the BVP at time $t = t_{n+1}$:

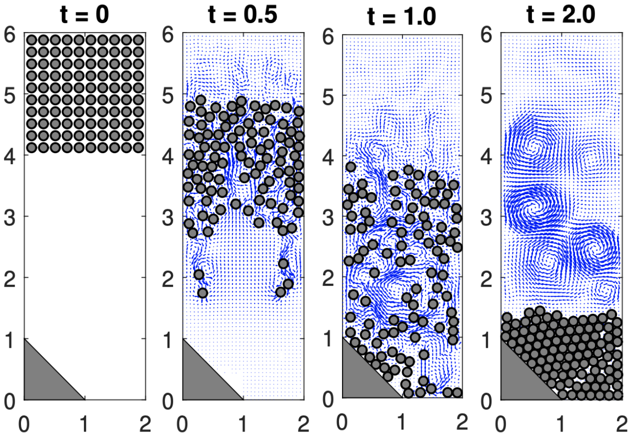
$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} + \boldsymbol{F}^{n+1} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega. \end{aligned}$$

- *It is highly inefficient in solving this BVP directly, even if $\boldsymbol{F}^{n+1}$ is already known. This is the reason for proposing the projection approach to decouple the computation of $(\boldsymbol{u}^{n+1}, p^{n+1})$.*
- *Next, we will consider a direct-forcing IB approach based on the first-order Chorin projection scheme. The virtual force $\boldsymbol{F}^{n+1}$ will be specified in the scheme when we decouple the time-discretized problem.*

Flow past a swimming fish-like solid body



Sedimentation of multiple particles



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3. T.-L. Horng, P.-W. Hsieh, S.-Y. Yang, and C.-S. You, A simple direct-forcing immersed boundary projection method with prediction-correction for fluid-solid interaction problems, *Computers & Fluids,* 176 (2018), pp. 135-152.