# MA 7121: Topics in Scientific Computing I
# Principal Component Pursuit Problem

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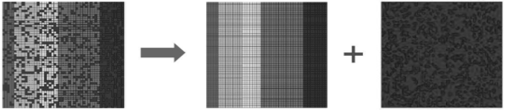
# Sparse plus low rank matrix decomposition

Let $\boldsymbol{M} \in \mathbb{R}^{m \times n}$ be a given grayscale image. Suppose that $\boldsymbol{M}$ is the superposition of a low-rank component $\boldsymbol{L}$ and a sparse component $\boldsymbol{S}$,

$$\boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S}.$$

We are interested in finding the low-rank image $\boldsymbol{L}$, which has high repeatability along horizontal or vertical directions.



(schematic diagram)

*The sparse plus low rank decomposition problem* can be formulated as the constrained minimization problem:

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left(\operatorname{rank}(\boldsymbol{L}) + \lambda \|\boldsymbol{S}\|_0\right) \quad \text{subject to} \quad \boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S},$$

where $\lambda > 0$ is a tuning parameter and $\|\boldsymbol{S}\|_0$ denotes the number of non-zero entries in $\boldsymbol{S}$. *The problem is not convex.*

# The principal component pursuit problem

We approximate the sparse plus low rank decomposition problem by the following *principal component pursuit (PCP) problem:*

$$\min_{\boldsymbol{L},\boldsymbol{S}} \left( \|\boldsymbol{L}\|_* + \lambda \|\boldsymbol{S}\|_1 \right) \quad \text{subject to} \quad \boldsymbol{M} = \boldsymbol{L} + \boldsymbol{S},$$

where $\|\boldsymbol{L}\|_*$ is the nuclear (Ky Fan/樊(土畿)) norm of $\boldsymbol{L}$ defined as

$$\|\boldsymbol{L}\|_* := \sum_{i=1}^{r} \sigma_i,$$

and $r \in \mathbb{N}^+$ is the rank of $\boldsymbol{L}$ and $\sigma_i$ are the singular values of $\boldsymbol{L}$, and $\|\boldsymbol{S}\|_1$ denotes the $\ell^1$-norm of $\boldsymbol{S}$ *(seen as a long vector in* $\mathbb{R}^{mn}$*),*

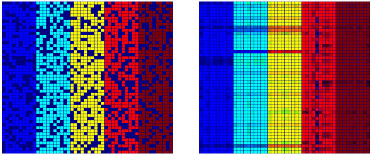
$$\|\boldsymbol{S}\|_1 := \sum_{i,j} |\boldsymbol{S}_{ij}|.$$

$\star$ *How about the existence of solution for the PCP problem?*
(cf. Candès-Li-Ma-Wright, J. ACM, 2011)

# Background recovering using the ALM method

$(\lambda, \mu) = (0.0007, 0.5)$

$(\lambda, \mu) = (0.006, 5)$



$(\lambda, \mu) = (0.007525, 0.04)$

$(\lambda, \mu) = (0.0025, 1.5)$

# References


1. E. J. Candès, X. Li, Y. Ma, and J. Wright, Robust principal component analysis? *Journal of the ACM,* 58 (2011), Article 11.
2. X. Ren and Z. Lin, Linearized alternating direction method with adaptive penalty and warm starts for fast solving transform invariant low-rank textures, *International Journal of Computer Vision,* 104, (2013), pp.1-14.