Direct-forcing immersed boundary projection methods for fluid-solid interaction problems

Suh-Yuh Yang (楊肅煜)

Department of Mathematics, National Central University
Jhongli District, Taoyuan City 320317, Taiwan

First version: August 22, 2018/Last updated: March 12, 2024

Outline of the lectures

1. Models of viscous incompressible fluid flow
   - Mass conservation and momentum conservation
   - Incompressible Euler equations for ideal fluids
   - Incompressible Navier-Stokes equations
2. Projection methods for incompressible Navier-Stokes equations
   - Helmholtz-Hodge decomposition
   - Chorin's first-order in time projection method
   - Second-order in time projection methods
3. Direct-forcing IB projection methods for FSI problems with or without prescribed solid velocity
   - A primitive direct-forcing IB projection method
   - A two-stage direct-forcing IB projection method
   - Equations of motion for non-prescribed solid velocity

Mass conservation

We consider a fluid of density ρ moving in a bounded region $\Omega \subset \mathbb{R}^3$ with velocity $\boldsymbol{u} = (u_1, u_2, u_3)^\top$. For a particular fixed, closed surface ∂D enclosing a volume $D \subseteq \Omega$, with the unit outward normal vector $\boldsymbol{n}$ to ∂D. Then we have

- **Mass conservation:** The rate of change of mass in D equals the amount of fluid flowing into D cross ∂D, i.e.,

$$\frac{d}{dt} \int_D \rho \, dV = - \int_{\partial D} \rho (\boldsymbol{u} \cdot \boldsymbol{n}) \, dS.$$

- **Divergence Theorem:** For a smooth vector field $\boldsymbol{u}$ on a bounded region D with a smooth boundary ∂D, we have

$$\int_{\partial D} \boldsymbol{u} \cdot \boldsymbol{n} \, dS = \int_D \nabla \cdot \boldsymbol{u} \, dV.$$

The incompressibility equation

According to the mass conservation, we have

$$\frac{d}{dt}\int_D \rho\, dV + \int_{\partial D} (\rho \boldsymbol{u}) \cdot \boldsymbol{n}\, dS = 0.$$

Furthermore, from the Divergence Theorem, we can get

$$0 = \frac{d}{dt}\int_D \rho\, dV + \int_{\partial D} (\rho \boldsymbol{u}) \cdot \boldsymbol{n}\, dS = \int_D \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u})\, dV.$$

Since D is an arbitrary chosen region in Ω, we can conclude that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \quad \text{in } \Omega.$$

Moreover, for incompressible and homogeneous fluid, the density ρ is constant with respect to both time and spatial coordinates, so we have

$$\nabla \cdot \boldsymbol{u} := \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0 \quad \text{in } \Omega.$$

Velocity of the fluid flow

Let us consider a particular small volume of fluid at $(\boldsymbol{x}, t)$.

- **Velocity:** If in a small interval of time δt, this small volume moves to position $\boldsymbol{x} + \delta \boldsymbol{x}$, then the velocity at position $\boldsymbol{x}$ and time t is given by

$$\boldsymbol{u} = (u_1, u_2, u_3)^\top := \lim_{\delta t \to 0} \frac{\delta \boldsymbol{x}}{\delta t}.$$

- **Taylor expansion:** The velocity depends on both position and time, so we write $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}, t)$ and $\boldsymbol{u} + \delta \boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x} + \delta \boldsymbol{x}, t + \delta t)$. By the Taylor Theorem, we have

$$\boldsymbol{q}(\boldsymbol{x} + \delta \boldsymbol{x}, t + \delta t) = \boldsymbol{q}(\boldsymbol{x}, t + \delta t) + (\delta \boldsymbol{x} \cdot \nabla)\boldsymbol{q}(\boldsymbol{x}, t + \delta t) + O(\|\delta \boldsymbol{x}\|^2),$$

$$\boldsymbol{q}(\boldsymbol{x}, t + \delta t) = \boldsymbol{q}(\boldsymbol{x}, t) + \delta t \frac{\partial}{\partial t}\boldsymbol{q}(\boldsymbol{x}, t) + O(\delta t^2).$$

Acceleration of the fluid flow

Since $\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x}, t)$ and $\boldsymbol{u} + \delta\boldsymbol{u} = \boldsymbol{q}(\boldsymbol{x} + \delta\boldsymbol{x}, t + \delta t)$, we have

$$\begin{aligned} \delta\boldsymbol{u} &= \boldsymbol{q}(\boldsymbol{x} + \delta\boldsymbol{x}, t + \delta t) - \boldsymbol{q}(\boldsymbol{x}, t) \\ &= \big(\boldsymbol{q}(\boldsymbol{x} + \delta\boldsymbol{x}, t + \delta t) - \boldsymbol{q}(\boldsymbol{x}, t + \delta t)\big) + \big(\boldsymbol{q}(\boldsymbol{x}, t + \delta t) - \boldsymbol{q}(\boldsymbol{x}, t)\big) \\ &= (\delta\boldsymbol{x} \cdot \nabla)\boldsymbol{q}(\boldsymbol{x}, t + \delta t) + \delta t \frac{\partial}{\partial t}\boldsymbol{q}(\boldsymbol{x}, t) + \big(O(\|\delta\boldsymbol{x}\|^2) + O(\delta t^2)\big). \end{aligned}$$

Hence, the acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$ of the fluid flow is given by

$$\begin{aligned} \boldsymbol{a} &= \frac{d\boldsymbol{u}}{dt} = \lim_{\delta t \to 0} \frac{\delta\boldsymbol{u}}{\delta t} \\ &= \lim_{\delta t \to 0} \left\{ \frac{(\delta\boldsymbol{x} \cdot \nabla)\boldsymbol{q}(\boldsymbol{x}, t + \delta t) + \delta t \frac{\partial}{\partial t}\boldsymbol{q}(\boldsymbol{x}, t)}{\delta t} + \frac{O(\|\delta\boldsymbol{x}\|^2) + O(\delta t^2)}{\delta t} \right\} \\ &= \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \end{aligned}$$

where we have used $\frac{\|\delta\boldsymbol{x}\|}{\delta t} \to \|\boldsymbol{u}\|$ and $\|\delta\boldsymbol{x}\| \to 0$ as $\delta t \to 0$.

Momentum conservation of the fluid flow

- **Momentum = mass × velocity:** $\boldsymbol{P} = m\,\boldsymbol{u}$, where m is the mass of the small volume at $(\boldsymbol{x}, t)$.
- **Momentum conservation:** If the external force $\boldsymbol{F}$ doesn't act on the system or the fluid, the momentum will not change by the time,

$$\frac{d\boldsymbol{P}}{dt} = \mathbf{0} \quad \text{if } \boldsymbol{F} = \mathbf{0}, \qquad \frac{d\boldsymbol{P}}{dt} \neq \mathbf{0} \quad \text{if } \boldsymbol{F} \neq \mathbf{0}.$$

By Newton's second law of motion, we know that the rate of change of momentum equals the external force acting on the fluid. We have

$$\boldsymbol{F}(\boldsymbol{x}, t) = \frac{d\boldsymbol{P}}{dt} = \frac{d(m\boldsymbol{u})}{dt} = m\boldsymbol{a} = m\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right).$$

Note that the acceleration $\boldsymbol{a}(\boldsymbol{x}, t) = \lim_{\delta t \to 0} \frac{\delta \boldsymbol{u}}{\delta t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}.$

The force on the ideal fluid

- An ideal fluid is an incompressible and homogeneous fluid that *has no viscosity,* so the only forces are due to the pressure (p) and the external body force such as gravity ($\boldsymbol{F}$). We define *the density of body force at the small volume of fluid at* $(\boldsymbol{x}, t)$ *as* $\boldsymbol{f}(\boldsymbol{x}, t) := \boldsymbol{F}/m$.
- The total force acting on the fluid contained in D is the pressure of the surrounding fluid plus the effect of the body force:

$$\textit{total force} = \int_{\partial D} p(-\boldsymbol{n})\, dS + \int_D \rho \boldsymbol{f}\, dV.$$

- According to Newton's second law of motion, we have

$$\int_D \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right) dV = \int_{\partial D} p(-\boldsymbol{n})\, dS + \int_D \rho \boldsymbol{f}\, dV.$$

Moving all of the terms to the left hand side, we have

$$\int_D \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \right) dV + \int_{\partial D} p\boldsymbol{n}\, dS - \int_D \rho \boldsymbol{f}\, dV = \boldsymbol{0}.$$

Converting the pressure term by an identity

In 3-D, let $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})^\top$ and $\boldsymbol{c} = (c_1, c_2, c_3)^\top$, then we have

$$\begin{aligned}
\nabla \cdot (p\boldsymbol{c}) &= \frac{\partial(pc_1)}{\partial x} + \frac{\partial(pc_2)}{\partial y} + \frac{\partial(pc_3)}{\partial z} \\
&= (c_1 \frac{\partial p}{\partial x} + p\frac{\partial c_1}{\partial x}) + (c_2 \frac{\partial p}{\partial y} + p\frac{\partial c_2}{\partial y}) + (c_3 \frac{\partial p}{\partial z} + p\frac{\partial c_3}{\partial z}) \\
&= (c_1 \frac{\partial p}{\partial x} + c_2 \frac{\partial p}{\partial y} + c_3 \frac{\partial p}{\partial z}) + (p\frac{\partial c_1}{\partial x} + p\frac{\partial c_2}{\partial y} + p\frac{\partial c_3}{\partial z}) \\
&= \boldsymbol{c} \cdot \nabla p + p \nabla \cdot \boldsymbol{c}.
\end{aligned}$$

Taking $\boldsymbol{v} = p\boldsymbol{c}$ in the Divergence Theorem, with $\boldsymbol{c}$ being a constant vector pointing in an arbitrary direction, we have

$$\int_{\partial D} (p\boldsymbol{c}) \cdot \boldsymbol{n}\, dS = \int_D \nabla \cdot (p\boldsymbol{c})\, dV = \int_D (\boldsymbol{c} \cdot \nabla p + p\nabla \cdot \boldsymbol{c})\, dV = \int_D \boldsymbol{c} \cdot \nabla p\, dV.$$

Since $\boldsymbol{c}$ *is an arbitrary constant vector,* so we can get

$$\int_{\partial D} p\boldsymbol{n}\, dS = \int_D \nabla p\, dV.$$

Euler equations for an ideal incompressible fluid

Since

$$\int_{\partial D} p\boldsymbol{n}\, dS = \int_D \nabla p\, dV,$$

we have

$$\int_D \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) + \nabla p - \rho\boldsymbol{f}\, dV = \boldsymbol{0}.$$

Note that D is an arbitrary chosen region in Ω. This leads to

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) + \nabla p - \rho\boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega.$$

Combining above equation with the mass conservation equation for incompressible and homogeneous fluid which *has no viscosity,* we obtain *the system of Euler equations for ideal incompressible fluids:*

$$\begin{cases} \rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) + \nabla p = \rho\boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega. \end{cases}$$

The normal stresses

- For a real viscous fluid, each small volume of fluid is not only acted on by *pressure forces (normal stresses)*, but also by *tangential stresses (shear stresses)*, i.e., the fluid not only acted on by pressure, but also by viscous stress.
- Thus, as in the inviscid case, the normal stresses are due to pressure giving rise to a force on the volume D of fluid,

$$\int_{\partial D} -p\boldsymbol{I}\boldsymbol{n}\, dS = \int_{\partial D} -p\boldsymbol{n}\, dS = \int_{D} -\nabla p\, dV,$$

where the pressure term is converted into an integral over D, exactly as we have done in the case of an ideal fluid.

The strain rate tensor

- The strain rate tensor (deformation tensor) is a physical quantity that describes the rate of change of the deformation of a material in the neighborhood of a certain point at a certain moment of time. It can be defined as the derivative of the strain tensor w.r.t. time, or as the symmetric part of the gradient (derivative w.r.t. position) of the flow velocity. *The strain rate tensor* is given by

$$\boldsymbol{\epsilon} := \frac{1}{2}\left(\left[\frac{\partial u_i}{\partial x_j}\right] + \left[\frac{\partial u_j}{\partial x_i}\right]\right) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top\right).$$

- We also define *the vorticity tensor* by

$$\boldsymbol{\xi} := \frac{1}{2}\left(\left[\frac{\partial u_i}{\partial x_j}\right] - \left[\frac{\partial u_j}{\partial x_i}\right]\right) = \frac{1}{2}\left(\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^\top\right).$$

- *gradient of the flow velocity = strain rate tensor + vorticity tensor*

$$\nabla \boldsymbol{u} = \boldsymbol{\epsilon} + \boldsymbol{\xi}.$$

The shear stress tensor

- Fluid satisfying a linear stress-strain relationship is called *a Newtonian fluid;* otherwise, the fluid is called *a non-Newtonian fluid.* In Newtonian fluid, the shear stress tensor $\boldsymbol{T}$ is a linear function of the strain rate tensor $\boldsymbol{\epsilon}$, defined by two coefficients, one relating to the expansion rate (the bulk viscosity coefficient) and one relating to the shear rate (the viscosity coefficient), i.e.,

$$\boldsymbol{T}(\boldsymbol{\epsilon}) = 2\mu\boldsymbol{\epsilon} + \lambda \, \text{trace}(\boldsymbol{\epsilon})\boldsymbol{I},$$

μ and λ are parameters describing the "stickiness" of the fluid.

- For an incompressible fluid, the parameter λ is not important because

$$\text{trace}(\boldsymbol{\epsilon}) = \nabla \cdot \boldsymbol{u} = 0.$$

Hence, we have the shear stress tensor for incompressible fluid,

$$\boldsymbol{T} = 2\mu\boldsymbol{\epsilon}.$$

An identity of $\nabla \cdot \epsilon$

$$
\begin{aligned}
\nabla \cdot \epsilon &= \nabla \cdot \frac{1}{2}\begin{pmatrix} 2\frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x} & \frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x} \\ \frac{\partial u_2}{\partial x}+\frac{\partial u_1}{\partial y} & 2\frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z}+\frac{\partial u_3}{\partial y} \\ \frac{\partial u_3}{\partial x}+\frac{\partial u_1}{\partial z} & \frac{\partial u_3}{\partial y}+\frac{\partial u_2}{\partial z} & 2\frac{\partial u_3}{\partial z} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial^2 u_1}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_2}{\partial y\partial x}+\frac{1}{2}\frac{\partial^2 u_1}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z\partial x}+\frac{1}{2}\frac{\partial^2 u_1}{\partial z^2} \\ \frac{1}{2}\frac{\partial^2 u_1}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z\partial y}+\frac{1}{2}\frac{\partial^2 u_2}{\partial z^2} \\ \frac{1}{2}\frac{\partial^2 u_1}{\partial x\partial z}+\frac{1}{2}\frac{\partial^2 u_3}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_2}{\partial y\partial z}+\frac{1}{2}\frac{\partial^2 u_3}{\partial y^2}+\frac{\partial^2 u_3}{\partial z^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}\frac{\partial^2 u_1}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_1}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u_1}{\partial z^2}+\frac{1}{2}\frac{\partial^2 u_1}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_2}{\partial y\partial x}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z\partial x} \\ \frac{1}{2}\frac{\partial^2 u_2}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_2}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u_2}{\partial z^2}+\frac{1}{2}\frac{\partial^2 u_1}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 u_2}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z\partial y} \\ \frac{1}{2}\frac{\partial^2 u_3}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u_3}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z^2}+\frac{1}{2}\frac{\partial^2 u_1}{\partial x\partial z}+\frac{1}{2}\frac{\partial^2 u_2}{\partial y\partial z}+\frac{1}{2}\frac{\partial^2 u_3}{\partial z^2} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \nabla^2 u_1 + \frac{\partial}{\partial x}(\nabla \cdot \boldsymbol{u}) \\ \nabla^2 u_2 + \frac{\partial}{\partial y}(\nabla \cdot \boldsymbol{u}) \\ \nabla^2 u_3 + \frac{\partial}{\partial z}(\nabla \cdot \boldsymbol{u}) \end{pmatrix}.
\end{aligned}
$$

Viscous shear forces

The molecular viscosity μ (the dynamic viscosity), which is a fluid property measuring the resistance of the fluid to shearing, gives rise to the viscous shear force

$$\int_{\partial D} \boldsymbol{T}\boldsymbol{n}\, dS = \int_D \nabla \cdot \boldsymbol{T}\, dV,$$

where we have used the Divergence Theorem, and

$$\begin{aligned} \nabla \cdot \boldsymbol{T} &= 2\mu \nabla \cdot \boldsymbol{\epsilon} = \mu \begin{pmatrix} \nabla^2 u_1 + \frac{\partial}{\partial x}(\nabla \cdot \boldsymbol{u}) \\ \nabla^2 u_2 + \frac{\partial}{\partial y}(\nabla \cdot \boldsymbol{u}) \\ \nabla^2 u_3 + \frac{\partial}{\partial z}(\nabla \cdot \boldsymbol{u}) \end{pmatrix} = \mu \begin{pmatrix} \nabla^2 u_1 \\ \nabla^2 u_2 \\ \nabla^2 u_3 \end{pmatrix} \\ &= \mu \nabla^2 \boldsymbol{u}, \end{aligned}$$

by virtue of the incompressibility condition, $\nabla \cdot \boldsymbol{u} = 0$. Thus, *the force due to the shear stresses* is given by

$$\int_{\partial D} \boldsymbol{T}\boldsymbol{n}\, dS = \int_D \nabla \cdot \boldsymbol{T}\, dV = \int_D \mu \nabla^2 \boldsymbol{u}\, dV.$$

Viscous fluid

Application of Newton's second law of motion in the case of a Newtonian fluid gives

$$\int_D \rho\left(\frac{d\boldsymbol{u}}{dt} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) dV = \int_{\partial D} (-p\boldsymbol{I} + \boldsymbol{T}(\boldsymbol{\epsilon}))\boldsymbol{n}\, dS + \int_D \rho\boldsymbol{f}\, dV.$$

Substituting

$$\int_{\partial D} -p\boldsymbol{I}\boldsymbol{n}\, dS = \int_D -\nabla p\, dV \quad \text{and} \quad \int_{\partial D} \boldsymbol{T}(\boldsymbol{\epsilon})\boldsymbol{n}\, dS = \int_D \mu\nabla^2\boldsymbol{u}\, dV$$

into the above equation, we have

$$\int_D \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) dV = \int_D \left(-\nabla p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f}\right) dV.$$

Using the fact that *$D \subseteq \Omega$ is an arbitrary region in the flow,* we finally obtain the incompressible Navier-Stokes equations.

The incompressible Navier-Stokes equations

The non-steady incompressible Navier-Stokes equations can be posed as

$$\begin{cases} \rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) &= -\nabla p + \mu\nabla^2\boldsymbol{u} + \rho\boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \end{cases}$$

which combined with boundary condition and initial data are the basis of practical models of incompressible viscous fluid flow.

Let $\nu := \mu/\rho$, called kinematic viscosity, and $p \leftarrow p/\rho$. The momentum equation can be simplified as

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\nabla^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega.$$

Time-dependent incompressible Navier-Stokes equations

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) and $[0, T]$ be the time interval. The time-dependent, incompressible Navier-Stokes problem can be posed as: find $\boldsymbol{u}$ and p with $\int_\Omega p\, dV = 0$, so that

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega \times (0, T], \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t = 0\}.
\end{aligned}$$

- $\boldsymbol{u}$ is the velocity field, p the pressure (divided by a constant density ρ), ν the kinematic viscosity, and $\boldsymbol{f}$ the body force.
- *By the Divergence Theorem, boundary velocity $\boldsymbol{u}_b$ must satisfy*

$$\int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n}\, dS = \int_\Omega \nabla \cdot \boldsymbol{u}\, dV = 0, \quad \forall\, t \in [0, T].$$

Time-discretization of the incompressible N-S equations

First, we discretize the time variable of the Navier-Stokes problem, with the spatial variable being left continuous. Consider the implicit Euler time-discretization with explicit first-order approximation to the nonlinear convection term:

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t} - \nu\nabla^2\boldsymbol{u}^{n+1} + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\
\nabla\cdot\boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $t_i := i\Delta t$ for $i = 0, 1, \cdots$, $\Delta t > 0$ is the time step length, and $\boldsymbol{g}^n$ denotes an approximate (or exact) value of $\boldsymbol{g}(t_n)$ at the time level n.

It is highly inefficient in solving this coupled system of Stokes-like equations directly. This is precisely the reason for proposing the projection approach to decouple the computation of $(\boldsymbol{u}^{n+1}, p^{n+1})$.

Idea of projection method

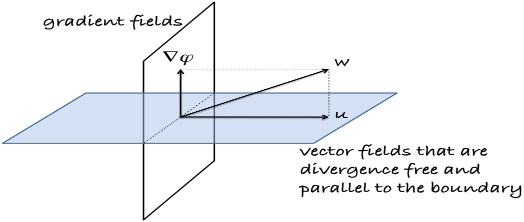
- This approach solves the equations of time-discretization of the incompressible Navier-Stokes equations.
- The underlying idea of projection method, first introduced by Chorin (1968, 1969) and Temam (1969), is based on applying the *Helmholtz-Hodge decomposition (HHD)* to the time-discretized incompressible Navier-Stokes equations.
- The feature of projection method is to compute velocity and pressure fields separately through the computation of an *intermediate velocity* $\boldsymbol{u}^*$, and then project it onto the space of divergence-free vector fields.

Helmholtz-Hodge decomposition (Chorin & Marsden's book)

A smooth vector field $\boldsymbol{w}$ defined on $\overline{\Omega}$ can be uniquely decomposed orthogonally in the form:

$$\boldsymbol{w} = \boldsymbol{u} + \nabla\varphi,$$

where $\boldsymbol{u}$ has zero divergence, $\nabla \cdot \boldsymbol{u} = 0$ in Ω, and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.



Remarks:

- Orthogonality means $\int_\Omega \boldsymbol{u} \cdot \nabla\varphi \, dV = 0$ (L^2-inner product).
- *The HHD describes the decomposition of a flow field $\boldsymbol{w}$ into its divergence-free component $\boldsymbol{u}$ and curl-free component $\nabla\varphi$, since $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \times (\nabla\varphi) = 0$ in Ω.*

Proof of HHD Theorem

Orthogonality of $\boldsymbol{u}$ and $\nabla\varphi$: First, note that

$$\nabla\cdot(\varphi\boldsymbol{u}) = (\nabla\cdot\boldsymbol{u})\varphi + \boldsymbol{u}\cdot\nabla\varphi.$$

Then by $\nabla\cdot\boldsymbol{u} = 0$ in Ω, Divergence Theorem, and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$,

$$\int_\Omega \boldsymbol{u}\cdot\nabla\varphi\,dV = \int_\Omega \nabla\cdot(\varphi\boldsymbol{u})\,dV = \int_{\partial\Omega}\varphi\boldsymbol{u}\cdot\boldsymbol{n}\,dS = 0. \quad (\clubsuit)$$

Uniqueness: Suppose $\boldsymbol{w} = \boldsymbol{u}_i + \nabla\varphi_i$, $\nabla\cdot\boldsymbol{u}_i = 0$ in Ω and $\boldsymbol{u}_i\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ for $i = 1, 2$. Then

$$(\boldsymbol{u}_1 - \boldsymbol{u}_2) + \nabla(\varphi_1 - \varphi_2) = 0 \quad \text{in } \Omega.$$

Taking the inner product with $\boldsymbol{u}_1 - \boldsymbol{u}_2$, we have

$$\begin{aligned} 0 &= \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{u}_1 - \boldsymbol{u}_2) + (\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot\nabla(\varphi_1 - \varphi_2)\,dV \\ &= \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2)\cdot(\boldsymbol{u}_1 - \boldsymbol{u}_2) + 0\,dV. \quad \text{(using } (\clubsuit) \text{ again)} \end{aligned}$$

It follows that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and $\nabla\varphi_1 = \nabla\varphi_2$.

Proof of HHD Theorem (cont.)

Existence: Given a smooth vector field $\boldsymbol{w}$, let φ be defined as the solution to the Neumann problem

$$\begin{cases} \nabla^2 \varphi = \nabla \cdot \boldsymbol{w} & \text{in } \Omega, \\ \nabla \varphi \cdot \boldsymbol{n} = \boldsymbol{w} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

It is known that the solution φ of this problem exists and is defined up to an arbitrary additive constant, see the Remark below. Define $\boldsymbol{u} := \boldsymbol{w} - \nabla \varphi$, then it is obvious that $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

Remark Consider the Neumann problem on a smooth domain D,

$$\begin{cases} \nabla^2 \psi = f & \text{in } D, \\ \nabla \psi \cdot \boldsymbol{n} = g & \text{on } \partial D. \end{cases}$$

The problem has a unique solution up to a constant if and only if the following compatibility condition holds:

$$\int_D f\, dV = \int_D \nabla \cdot \nabla \psi\, dV = \int_{\partial D} \nabla \psi \cdot \boldsymbol{n}\, dS = \int_{\partial D} g\, dS.$$

Chorin projection method (Math. Comput. 1968 & 1969)

Step 1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 2 Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Notice that Step 2 is equivalent to solving the following pressure-Poisson equation with the homogeneous Neumann boundary condition:

$$\begin{cases} \nabla^2 p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \boldsymbol{u}^* & \text{in } \Omega, \\ \nabla p^{n+1} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and then define the velocity field by $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1}$.

Remarks on Chorin's first-order method

1. The second step is usually referred to as the projection step.

$$\boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \Delta t \nabla p^{n+1} = \boldsymbol{u}^{n+1} + \nabla(\Delta t p^{n+1}).$$

This is indeed the standard HHD of $\boldsymbol{u}^*$ when $\boldsymbol{u}_b^{n+1} = \mathbf{0}$ on $\partial\Omega$.

2. Summing all equations in Chorin's projection method, we have

$$\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} &= \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega, \end{aligned}$$

different from the original semi-implicit discretization. Since

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1} \approx \boldsymbol{u}^* \text{ in } \Omega \text{ as } \Delta t \to 0^+,$$

it is not surprising that we should expect

$$\nabla^2 \boldsymbol{u}^{n+1} \approx \nabla^2 \boldsymbol{u}^* \text{ in } \Omega \text{ and } \boldsymbol{u}^{n+1} \approx \boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega \text{ as } \Delta t \to 0^+.$$

Error estimates of Chorin's first-order method

- *The boundary condition* $\nabla p^{n+1} \cdot \boldsymbol{n} = 0$ *on* $\partial\Omega$ *is enforce on pressure.* Rannacher (1991) showed that this artificial Neumann boundary condition induces a numerical boundary layer on the pressure.
- (Prohl 1997, Rannacher 1991, Shen 1992) Assuming that $(\boldsymbol{u}^e, p^e)$, solving the Stokes equations, is sufficiently smooth. Then the solution of above projection method satisfies the error estimates:

$$\|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} + \|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}^*_{\Delta t}\|_{\ell^\infty([L^2(\Omega)]^d)} \le c(\boldsymbol{u}^e, p^e, T)\Delta t,$$
$$\|p^e_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \|\boldsymbol{u}^e_{\Delta t} - \boldsymbol{u}^*_{\Delta t}\|_{\ell^\infty([H^1(\Omega)]^d)} \le c(\boldsymbol{u}^e, p^e, T)\sqrt{\Delta t},$$

where $\varphi_{\Delta t} = \{\varphi^0, \varphi^1, \cdots, \varphi^N\}$ denotes some sequence of functions in a Hilbert space $\mathcal{H}$ and define the discrete norm:

$$\|\varphi_{\Delta t}\|_{\ell^2(\mathcal{H})} := \left(\Delta t \sum_{k=1}^{N} \|\varphi^k\|^2_{\mathcal{H}}\right)^{1/2}, \ \|\varphi_{\Delta t}\|_{\ell^\infty(\mathcal{H})} := \max_{0\le k\le N} \|\varphi^k\|_{\mathcal{H}}.$$

Numerical boundary layer on the pressure

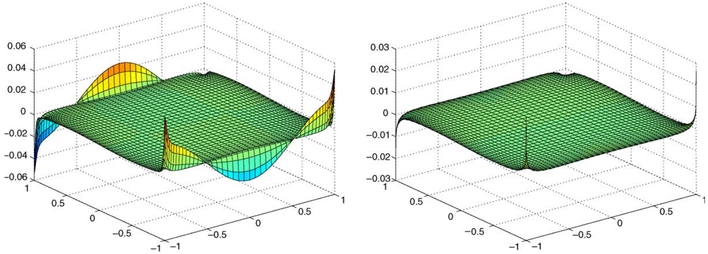


Fig. 1. Pressure error field at time $t = 1$ in a square: (left) standard form; (right) rotational form.

J. L. Guermond, P. Minev, and J. Shen, An overview of projection methods for incompressible flows, *Computer Methods in Applied Mechanics and Engineering*, 195 (2006), pp. 6011-6045.

Second-order time-discretization

Using the implicit second-order Crank-Nicolson formula, we have

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t} - \frac{\nu}{2}\nabla^2(\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n}) & \\
+[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + [\nabla p]^{n+\frac{1}{2}} &= [\boldsymbol{f}]^{n+\frac{1}{2}} \quad \text{in } \Omega, \\
\nabla\cdot\boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $[\boldsymbol{g}]^{n+\frac{1}{2}}$ denotes some explicit second-order approximation to $\frac{1}{2}(\boldsymbol{g}^{n+1}+\boldsymbol{g}^{n})$ or denotes the exact value. Two popular choices are:

$$
\frac{1}{2}\left(\boldsymbol{g}^{n+1}+\boldsymbol{g}^{n}\right) = \frac{3}{2}\boldsymbol{g}^{n} - \frac{1}{2}\boldsymbol{g}^{n-1} + O(\Delta t^2) \quad \text{(Adams-Bashforth)},
$$

$$
\frac{1}{2}\left(\boldsymbol{g}^{n+1}+\boldsymbol{g}^{n}\right) = \boldsymbol{g}^{n+\frac{1}{2}} + O(\Delta t^2) = 2\boldsymbol{g}^{n-\frac{1}{2}} - \boldsymbol{g}^{n-\frac{3}{2}} + O(\Delta t^2).
$$

Again, it is inefficient to solve the semi-implicit equations directly. We will solve it by using the projection approach.

Explicit second-order approximations

Consider the smooth scalar function g. By Taylor's expansion,

$$
\begin{aligned}
g(t_{n+1}) &= g(t_n) + \Delta t g'(t_n) + \tfrac{\Delta t^2}{2} g''(t_n) + \cdots && (1)\\
g(t_{n-1}) &= g(t_n) - \Delta t g'(t_n) + \tfrac{\Delta t^2}{2} g''(t_n) - \cdots && (2)\\
g(t_{n+\frac{1}{2}}) &= g(t_n) + \tfrac{\Delta t}{2} g'(t_n) + \tfrac{\Delta t^2}{8} g''(t_n) + \cdots && (3)
\end{aligned}
$$

Adding (1) to (2) and $2 \times (3) - (1)$, we obtain

$$
\begin{aligned}
g(t_{n+1}) + g(t_{n-1}) &= 2g(t_n) + \Delta t^2 g''(t_n) + \cdots && (\star)\\
2g(t_{n+\frac{1}{2}}) - g(t_{n+1}) &= g(t_n) - \tfrac{\Delta t^2}{4} g''(t_n) + \cdots && (\star\star)
\end{aligned}
$$

For example, by $(\star)$ and combining $(\star)$ with $(\star\star)$, we have

$$
\begin{aligned}
&\tfrac{1}{2}\left\{((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n+1} + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n}\right\}\\
&\quad = \tfrac{1}{2}\left\{2((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n} - ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n-1} + O(\Delta t^2) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n}\right\}\\
&\qquad = \tfrac{3}{2}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n} - \tfrac{1}{2}((\boldsymbol{u}\cdot\nabla)\boldsymbol{u})^{n-1} + O(\Delta t^2),
\end{aligned}
$$

$$
\tfrac{1}{2}\left(\nabla p^{n+1} + \nabla p^{n}\right) = \nabla p^{n+\frac{1}{2}} + O(\Delta t^2) = 2\nabla p^{n-\frac{1}{2}} - \nabla p^{n-\frac{3}{2}} + O(\Delta t^2).
$$

Bell-Colella-Glaz projection method (JCP 1989)

Step 1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\boldsymbol{u}^* + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} \text{ in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega. \end{cases}$$

Step 2 Determine $\boldsymbol{u}^{n+1}$ and φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

It is equivalent to solving the φ^{n+1}-Neumann Poisson problem:

$$\begin{cases} \nabla^2\varphi^{n+1} = \dfrac{1}{\Delta t}\nabla\cdot\boldsymbol{u}^* \quad \text{in } \Omega, \\ \nabla\varphi^{n+1}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and then set $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t\nabla\varphi^{n+1}$.

Step 3 Update the pressure, $p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \varphi^{n+1}$.

Brown-Cortez-Minion projection method (JCP 2001)

- The Bell-Colella-Glaz projection method produces solutions that converge at a second-order rate for the velocity, but the pressure converges at only a first-order rate.
- Brown-Cortez-Minion suggest that pressure correction equation in Step 3 should be modified as

Step 3 Update the pressure,

$$p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \varphi^{n+1} - \frac{\nu \Delta t}{2} \nabla^2 \varphi^{n+1}$$

that recovers second-order accuracy in the pressure. Moreover, they suggest the following general procedure to generate second-order accurate projection methods, see next two slides.

A general second-order projection method

Brown-Cortez-Minion (JCP 2001) suggested the general procedure:

Step 1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\boldsymbol{u}^* + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla q = & [\boldsymbol{f}]^{n+\frac{1}{2}} \quad \text{in } \Omega, \\ \mathcal{B}\boldsymbol{u}^* = & \boldsymbol{0} \quad \text{on } \partial\Omega, \end{cases}$$

where ∇q is a prediction of $\nabla p^{n+\frac{1}{2}}$ and $\mathcal{B}\boldsymbol{u}^* = \boldsymbol{0}$ is some appropriate boundary condition.

Step 2 Determine $\boldsymbol{u}^{n+1}$ and φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0 \quad \text{in } \Omega, \end{cases}$$

with some boundary condition of $\boldsymbol{u}^{n+1}$, consistent with $\mathcal{B}\boldsymbol{u}^* = \boldsymbol{0}$ and the expectation $\boldsymbol{u}^{n+1}|_{\partial\Omega} = \boldsymbol{u}_b^{n+1}$, and φ^{n+1} is an auxiliary function whose main purpose is to project $\boldsymbol{u}^*$.

Step 3 Update the pressure, $p^{n+\frac{1}{2}} = q + \mathcal{L}\varphi^{n+1}$, $\mathcal{L}$ is some operator needed to be determined.

Remarks on the general method

1. Three parts need to be made in the design of such a method:
   - the prediction ∇q of $\nabla p^{n+\frac{1}{2}}$
   - the boundary condition $\mathcal{B}\boldsymbol{u}^* = \mathbf{0}$ on $\partial\Omega$
   - the operator $\mathcal{L}$ in the pressure-update equation
2. Combining the Step 1 and Step 2 and eliminating $\boldsymbol{u}^*$, we have

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \frac{\nu}{2}\nabla^2(\boldsymbol{u}^{n+1} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}}$$
$$+\nabla\left(q + \varphi^{n+1} - \frac{\nu\Delta t}{2}\nabla^2\varphi^{n+1}\right) = [\boldsymbol{f}]^{n+\frac{1}{2}}.$$

Comparing the equation with original second-order semi-implicit time-discretization equation, we obtain

$$p^{n+\frac{1}{2}} = q + \varphi^{n+1} - \frac{\nu\Delta t}{2}\nabla^2\varphi^{n+1} \quad \left(= q + \varphi^{n+1} - \frac{\nu}{2}\nabla\cdot\boldsymbol{u}^*\right).$$

This means that the operator $\mathcal{L}$ should be defined as

$$\mathcal{L}\varphi^{n+1} := \varphi^{n+1} - \frac{\nu\Delta t}{2}\nabla^2\varphi^{n+1} \quad \left(= \varphi^{n+1} - \frac{\nu}{2}\nabla\cdot\boldsymbol{u}^*\right).$$

Kim-Moin projection method (JCP 1985)

Step 1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\boldsymbol{u}^* + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} & \text{in } \Omega, \\ \boldsymbol{u}^*\cdot\boldsymbol{n} = \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} \ \ \& \ \ \boldsymbol{u}^*\cdot\boldsymbol{\tau} = (\boldsymbol{u}_b^{n+1} + \Delta t\nabla\varphi^n)\cdot\boldsymbol{\tau} & \text{on } \partial\Omega. \end{cases}$$

Step 2 Determine $\boldsymbol{u}^{n+1}$ and $p^{n+\frac{1}{2}}$ by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

It is equivalent to solving the φ^{n+1}-Neumann Poisson problem:

$$\begin{cases} \nabla^2\varphi^{n+1} = \dfrac{1}{\Delta t}\nabla\cdot\boldsymbol{u}^* & \text{in } \Omega, \\ \nabla\varphi^{n+1}\cdot\boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and set $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t\nabla\varphi^{n+1}$. Moreover, we have $\boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n}$ and $\boldsymbol{u}^{n+1}\cdot\boldsymbol{\tau} = \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{\tau} - \Delta t\nabla(\varphi^{n+1} - \varphi^n)\cdot\boldsymbol{\tau}$.

Step 3 Update the pressure, $p^{n+\frac{1}{2}} = \varphi^{n+1} - \dfrac{\nu\Delta t}{2}\nabla^2\varphi^{n+1}$.

Kim-Moin and Brown-Cortez-Minion are "equivalent"!

Staring with the Kim-Moin method and changing the variables,

$$\begin{aligned}\widetilde{\boldsymbol{u}} &= \boldsymbol{u}^* - \Delta t \nabla \varphi^n, \qquad \psi^{n+1} = \varphi^{n+1} - \varphi^n,\\ p^{n+\frac{1}{2}} &= \varphi^{n+1} - \frac{\nu \Delta t}{2}\nabla^2 \varphi^{n+1}, \qquad p^{n-\frac{1}{2}} = \varphi^n - \frac{\nu\Delta t}{2}\nabla^2\varphi^n,\end{aligned}$$

we can find that solution $(\boldsymbol{u}^{n+1}, p^{n+\frac{1}{2}})$ of the Kim-Moin method also solves the Brown-Cortez-Minion projection equations and *vice versa*:

$$\left\{\begin{aligned} \frac{\widetilde{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} - \frac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} &= [\boldsymbol{f}]^{n+\frac{1}{2}} \quad \text{in } \Omega,\\ \widetilde{\boldsymbol{u}} &= \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega; \end{aligned}\right.$$

$$\left\{\begin{aligned} \frac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}}{\Delta t} + \nabla\psi^{n+1} &= \boldsymbol{0} \quad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{u}^{n+1} &= 0 \quad \text{in } \Omega,\\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} &= \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} \quad \text{on } \partial\Omega; \end{aligned}\right.$$

$$p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \psi^{n+1} - \frac{\nu\Delta t}{2}\nabla^2\psi^{n+1}.$$

Choi-Moin projection method (JCP 1994)

Step 1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} & \text{in } \Omega, \\ & \widetilde{\boldsymbol{u}} = \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega; \\ \dfrac{\boldsymbol{u}^* - \widetilde{\boldsymbol{u}}}{\Delta t} - \nabla p^{n-\frac{1}{2}} = \boldsymbol{0} \quad \text{in } \Omega. & \end{cases}$$

Step 2 Determine $\boldsymbol{u}^{n+1}$ and $p^{n+\frac{1}{2}}$ by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+\frac{1}{2}} & = & \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} & = & 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} & = & \boldsymbol{u}^*\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

It is equivalent to solving the $p^{n+\frac{1}{2}}$-Neumann Poisson problem:

$$\begin{cases} \nabla^2 p^{n+\frac{1}{2}} & = & \dfrac{1}{\Delta t}\nabla\cdot\boldsymbol{u}^* \quad \text{in } \Omega, \\ \nabla p^{n+\frac{1}{2}}\cdot\boldsymbol{n} & = & 0 \quad \text{on } \partial\Omega, \end{cases}$$

and then set $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* - \Delta t \nabla p^{n+\frac{1}{2}}$. Moreover, we have

$$\boldsymbol{u}^{n+1} = \widetilde{\boldsymbol{u}} - \Delta t(\nabla p^{n+\frac{1}{2}} - \nabla p^{n-\frac{1}{2}}) = \boldsymbol{u}_b^{n+1} + O(\Delta t^2) \quad \text{on } \partial\Omega.$$

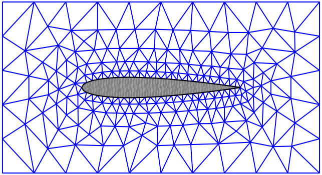
Remark on the error estimates

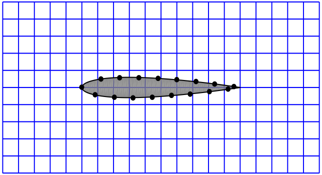
1. Brown-Cortez-Minion have showed that the pressure approximation is second-order accurate, but the numerical experiments reported show that this result is valid in a periodic channel only, and that *convergence rate of* $3/2$ *for the pressure is likely to be the best possible for general domain.*

2. J. L. Guermond, P. Minev, and J. Shen, An overview of projection methods for incompressible flows, *Computer Methods in Applied Mechanics and Engineering*, 195 (2006), pp. 6011-6045.

Fluid-structure interaction problem (流構耦合問題)

- A fluid-structure interaction (FSI) problem describes the coupled dynamics of fluid mechanics and structure mechanics. The study of FSI problems is of great importance in many sciences and engineering applications
- It usually requires the modeling of complex geometric structure and moving boundaries. It is very challenging for conventional body-fitted approach.
- In what follows, we adopt a Cartesian grid based non-boundary conforming method, *the direct-forcing immersed boundary (IB) projection method.*

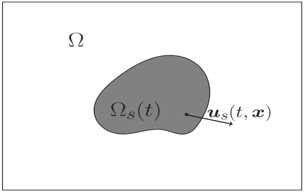




A fluid-solid interaction problem

Let Ω be the fluid domain which encloses a rigid body positioned at $\overline{\Omega}_s(t)$ *with a prescribed velocity* $\boldsymbol{u}_s(t, \boldsymbol{x})$. A typical problem is flow over a stationary or moving solid ball with prescribed velocity. The governing equations of the fluid-solid interaction problem with initial value and no-slip boundary condition can be posed as follows:

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times (0, T], \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times (0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_s \quad \text{on } \partial\Omega_s \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times \{t = 0\}.
\end{aligned}$$



The body-fitted approach

The body-fitted approach is a conventional method for solving the time-dependent, incompressible Navier-Stokes equations on a domain enclosing a rigid body. For example, using the implicit Euler time-discretization at time t_{n+1}, we solve the linearization in the spatial domain $\Omega \setminus \overline{\Omega}_s^{n+1}$,

$$
\begin{aligned}
\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^{n+1} + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} &= \boldsymbol{f}^{n+1} \text{ in } \Omega \setminus \overline{\Omega}_s^{n+1}, \\
\nabla \cdot \boldsymbol{u}^{n+1} &= 0 \text{ in } \Omega \setminus \overline{\Omega}_s^{n+1}, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega, \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}_s^{n+1} \text{ on } \partial\Omega_s^{n+1}.
\end{aligned}
$$

Again, it is highly inefficient in solving these equations directly. Below, we consider the direct-forcing immersed boundary approach.

A direct-forcing approach: virtual force F

A virtual force term $\boldsymbol{F}$ is added to the momentum equation to accommodate interaction between the solid and the fluid, and we expect the problem can be solved on the whole domain Ω and do not need to set the interior boundary condition $\boldsymbol{u}_s$ on the interface $\partial\Omega_s$:

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu\nabla^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} + \boldsymbol{F} \quad \text{in } \Omega \times (0,T], \\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega \times (0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0,T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } \Omega \times \{t=0\}.
\end{aligned}
$$

The virtual force $\mathbf{F}$ *exists in the rigid body* $\overline{\Omega}_s(t)$ *which is treated as a portion of the fluid but the virtual force enforces it to act like a solid body.*

The virtual force will be specified in the time-discrete equations when we apply the projection methods to solve the time-discretization problem. We first consider the first-order projection method of Chorin.

A primitive direct-forcing IB projection method (Chorin)

The main idea was proposed by Noor-Chern-Horng (CM 2009).

Step 1 Solve the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 2 Determine $\boldsymbol{u}^{**}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{**} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{**} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

It is equivalent to solving the p^{n+1}-Neumann Poisson problem:

$$\begin{cases} \nabla^2 p^{n+1} = \dfrac{1}{\Delta t} \nabla \cdot \boldsymbol{u}^* & \text{in } \Omega, \\ \nabla p^{n+1} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and set $\boldsymbol{u}^{**} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1} \Longrightarrow \nabla \cdot \boldsymbol{u}^{**} = 0, \boldsymbol{u}^{**} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n}$

A primitive direct-forcing IB projection method (Chorin)

Step 3 Define the virtual force $\boldsymbol{F}^{n+1}$ and then determine the velocity field $\boldsymbol{u}^{n+1}$ by setting

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{**}}{\Delta t} = \boldsymbol{F}^{n+1} := \eta \frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \Omega,$$

where $\eta(\boldsymbol{x}, t_{n+1})$ is defined by

$$\eta(\boldsymbol{x}, t_{n+1}) = \begin{cases} 1 & \boldsymbol{x} \in \overline{\Omega}_s^{n+1}, \\ 0 & \boldsymbol{x} \notin \overline{\Omega}_s^{n+1}. \end{cases}$$

The virtual force $\boldsymbol{F}^{n+1}$ exists on the whole solid body and zero elsewhere. In other words, in this step, we simply set

$$\boldsymbol{u}^{n+1} = \begin{cases} \boldsymbol{u}^{**} & \text{in } \overline{\Omega} \setminus \overline{\Omega}_s^{n+1}, \\ \boldsymbol{u}_s & \text{in } \overline{\Omega}_s^{n+1}. \end{cases}$$

We remark that η can be taken fractional on the boundary cells.

A primitive direct-forcing IB projection method (Brown et al.)

Step 1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\boldsymbol{u}^* + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} \text{ in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega. \end{cases}$$

Step 2 Determine $\boldsymbol{u}^{**}$ and φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \mathbf{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{**} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{**}\cdot\boldsymbol{n} = \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

Step 3 Define the virtual force $\boldsymbol{F}^{n+\frac{1}{2}}$ and then determine the velocity field $\boldsymbol{u}^{n+1}$ by setting

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{**}}{\Delta t} = \boldsymbol{F}^{n+\frac{1}{2}} := \eta\frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \Omega.$$

Step 4 Update the pressure, $p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \varphi^{n+1} - \dfrac{\nu\Delta t}{2}\nabla^2\varphi^{n+1}$.

Remarks on the direct-forcing IB projection method

1. Conventionally, Step 2 would be the end of projection method for velocity field and actually $\boldsymbol{u}^{n+1} = \boldsymbol{u}^{**}$. However, in order to satisfy the no-slip boundary condition at the interface $\partial\Omega_s^{n+1}$, we need Step 3 to reset the velocity to be the same as that of the solid's velocity $\boldsymbol{u}_s$.

2. Summing all equations in the direct-forcing IB method based on, e.g., the Chorin projection method, we obtain

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n + \nabla p^{n+1} = \boldsymbol{f}^{n+1} + \boldsymbol{F}^{n+1} \quad \text{in } \Omega,$$

$$\nabla \cdot \boldsymbol{u}^{**} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u}^{**} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega,$$

$$\boldsymbol{u}^{**} = \boldsymbol{u}^* - \Delta t \nabla p^{n+1} \quad \text{in } \Omega, \qquad \boldsymbol{u}^{n+1} = \begin{cases} \boldsymbol{u}^{**} & \text{in } \overline{\Omega} \setminus \overline{\Omega}_s^{n+1}, \\ \boldsymbol{u}_s & \text{in } \overline{\Omega}_s^{n+1}. \end{cases}$$

Is the above system "a good approximation" to the system of the body-fitted approach?

What's wrong with the direct-forcing IB projection method?

1. Although the direct-forcing IB projection method seems to produce reasonable results for many fluid-solid interaction problems, *it violates our physical intuition!*
2. It is not always stable when the direct-forcing IB approach combined with an arbitrary chosen projection method.

   The reason for this is because the velocity and pressure used in solving the intermediate velocity field $\boldsymbol{u}^*$ *may be not consistent!*

In what follows, we will propose a simple remedy to retrieve the direct-forcing IB projection method proposed by Noor-Chern-Horng.

We will use the idea of the prediction-correction approach to fit the physical intuition and carefully choose a "good" projection method!

A two-stage direct-forcing IB projection method (Chorin)

Prediction –

Step 1.1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\left\{\begin{array}{rcll} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu\nabla^2\boldsymbol{u}^* + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n & = & \boldsymbol{f}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* & = & \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{array}\right.$$

Step 1.2 Determine $\boldsymbol{u}^{**}$ and p^{n+1} by solving

$$\left\{\begin{array}{rcll} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} & = & \boldsymbol{0} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{**} & = & 0 & \text{in } \Omega, \\ \boldsymbol{u}^{**}\cdot\boldsymbol{n} & = & \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} & \text{on } \partial\Omega. \end{array}\right.$$

Step 1.3 Predict the virtual force $\widetilde{\boldsymbol{F}}^{n+1}$ that exits on the solid body and zero elsewhere by setting

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{**}}{\Delta t} = \widetilde{\boldsymbol{F}}^{n+1} := \eta\,\frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \Omega, \;\; \eta(\boldsymbol{x}, t_{n+1}) = \left\{\begin{array}{ll} 1 & \boldsymbol{x}\in\overline{\Omega}_s^{n+1}, \\ 0 & \boldsymbol{x}\notin\overline{\Omega}_s^{n+1}, \end{array}\right.$$

which implies that the velocity $\boldsymbol{u}^{n+1}$ in the solid $\overline{\Omega}_s^{n+1}$ is equal to $\boldsymbol{u}_s$.

A two-stage direct-forcing IB projection method (Chorin)

Correction –

Step 2.1 Solve the intermediate velocity field $\boldsymbol{u}^*$,

$$\left\{ \begin{array}{rcl} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n & = & \boldsymbol{f}^{n+1} + \widetilde{\boldsymbol{F}}^{n+1} \quad \text{in } \Omega, \\ \boldsymbol{u}^* & = & \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega. \end{array} \right.$$

Step 2.2 Determine $\boldsymbol{u}^{**}$ and p^{n+1} by solving

$$\left\{ \begin{array}{rcl} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} & = & \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{**} & = & 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{**} \cdot \boldsymbol{n} & = & \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega. \end{array} \right.$$

Step 2.3 Correct the velocity field $\boldsymbol{u}^{n+1}$ and virtual force $\boldsymbol{F}^{n+1}$,

$$\boldsymbol{u}^{n+1} = \left\{ \begin{array}{ll} \boldsymbol{u}^{**} & \text{in } \overline{\Omega} \setminus \overline{\Omega}_s^{n+1} \\ \boldsymbol{u}_s & \text{in } \overline{\Omega}_s^{n+1} \end{array} \right. \quad \text{and} \quad \boldsymbol{F}^{n+1} = \widetilde{\boldsymbol{F}}^{n+1} + \eta \frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \text{ in } \overline{\Omega}_s^{n+1}.$$

A two-stage direct-forcing IB projection method (Choi-Moin)

Prediction –

Step 1.1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \widetilde{\boldsymbol{u}} = \boldsymbol{u}_b^{n+1} \quad \text{on } \partial\Omega; \\ \dfrac{\boldsymbol{u}^* - \widetilde{\boldsymbol{u}}}{\Delta t} - \nabla p^{n-\frac{1}{2}} = \boldsymbol{0} \quad \text{in } \Omega. \end{cases}$$

Step 1.2 Determine $\boldsymbol{u}^{**}$ and φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{**} = 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{**}\cdot\boldsymbol{n} = \boldsymbol{u}^*\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

Step 1.3 Predict the virtual force $\widetilde{\boldsymbol{F}}^{n+\frac{1}{2}}$ by setting

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{**}}{\Delta t} = \widetilde{\boldsymbol{F}}^{n+\frac{1}{2}} := \eta\,\frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \Omega.$$

A two-stage direct-forcing IB projection method (Choi-Moin)

Correction –

Step 2.1 Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\widetilde{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} - \dfrac{\nu}{2}\nabla^2(\widetilde{\boldsymbol{u}} + \boldsymbol{u}^n) + [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]^{n+\frac{1}{2}} + \nabla p^{n-\frac{1}{2}} = [\boldsymbol{f}]^{n+\frac{1}{2}} + \widetilde{\boldsymbol{F}}^{n+\frac{1}{2}} \text{ in } \Omega, \\ \qquad \widetilde{\boldsymbol{u}} = \boldsymbol{u}_b^{n+1} \text{ on } \partial\Omega; \\ \dfrac{\boldsymbol{u}^* - \widetilde{\boldsymbol{u}}}{\Delta t} - \nabla p^{n-\frac{1}{2}} = \boldsymbol{0} \quad \text{in } \Omega. \end{cases}$$

Step 2.2 Determine $\boldsymbol{u}^{**}$ and correct φ^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla\varphi^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{**} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{**}\cdot\boldsymbol{n} = \boldsymbol{u}^*\cdot\boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Step 2.3 Correct the velocity $\boldsymbol{u}^{n+1}$ and virtual force $\boldsymbol{F}^{n+\frac{1}{2}}$,

$$\boldsymbol{u}^{n+1} = \begin{cases} \boldsymbol{u}^{**} & \text{in } \overline{\Omega}\setminus\overline{\Omega}_s^{n+1} \\ \boldsymbol{u}_s & \text{in } \overline{\Omega}_s^{n+1} \end{cases} \quad \text{and } \boldsymbol{F}^{n+\frac{1}{2}} = \widetilde{\boldsymbol{F}}^{n+\frac{1}{2}} + \eta\frac{\boldsymbol{u}_s - \boldsymbol{u}^{**}}{\Delta t} \quad \text{in } \overline{\Omega}_s^{n+1}.$$

Step 2.4 Update the pressure as $p^{n+\frac{1}{2}} = \varphi^{n+1} - \dfrac{\nu}{2}\nabla\cdot\widetilde{\boldsymbol{u}}$.

Space-discretization on a staggered grid

In following numerical experiments, we employ the two-stage direct-forcing IB projection method (based on Choi-Moin scheme).

- We apply the second-order centered differences over a staggered grid for space-discretization:

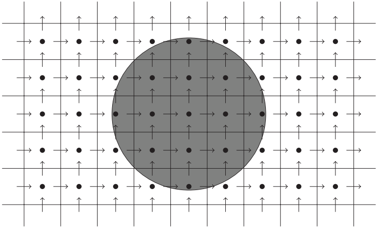


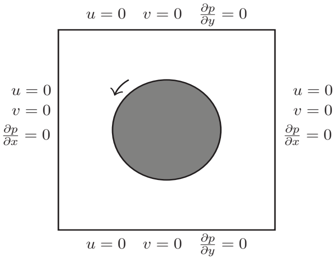
Diagram of the computational domain Ω with staggered grid, where the unknowns u, v and p are approximated at the grid points marked by $\rightarrow$, $\uparrow$ and $\bullet$, respectively

- In all examples, the body force $\boldsymbol{f}$ are zero. The volume-of-solid function η is fractional on the boundary cells.

Example 1: rotating solid disk

Problem setting –

- The computational domain is $\Omega = (0,1) \times (0,1)$, within which there is a rotating solid disk centered at $(0.5, 0.5)$ with radius 0.25. The disk rotates counterclockwise by a constant angular velocity $\omega = 4$.
- The Reynolds number is $Re := 1/\nu = 100$, time step length is $\Delta t = 0.1h$ (CFL number is 0.1), and $T = 4$.



The homogeneous boundary conditions on $\partial\Omega$

Example 1: error behavior

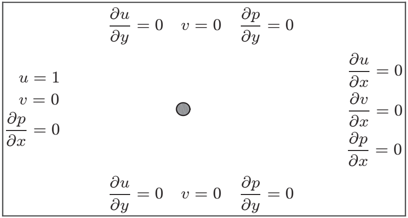
Error behavior of the numerical solutions u_h, v_h, and p_h at $T = 4$ using the solution of $h = 1/1620$ as the reference solution

	$1/h$	L^1 norm	order	L^2 norm	order	Max. norm	order
u_h	20	2.0820e-02	–	3.9742e-02	–	1.7573e-01	–
	60	8.4854e-03	0.82	1.7044e-02	0.77	8.2900e-02	0.68
	180	2.5123e-03	1.11	5.0608e-03	1.11	2.8370e-02	0.98
	540	6.5240e-04	1.23	1.3207e-03	1.22	8.1061e-03	1.14
v_h	20	2.5334e-02	–	4.2845e-02	–	1.7573e-01	–
	60	1.0199e-02	0.83	1.8496e-02	0.76	8.2900e-02	0.68
	180	3.0741e-03	1.09	5.5503e-03	1.10	2.8554e-02	0.97
	540	7.9659e-04	1.23	1.4500e-03	1.22	8.1061e-03	1.15
p_h	20	6.8326e-03	–	1.3968e-02	–	8.4475e-02	–
	60	3.0749e-03	0.73	6.2523e-03	0.73	4.8072e-02	0.51
	180	9.8066e-04	1.04	2.1771e-03	0.96	3.8831e-02	0.19
	540	2.6861e-04	1.18	7.5445e-04	0.96	2.5701e-02	0.38

Example 2: flow past a stationary cylinder

Problem setting –

- $\Omega = (-13.4D, 16.5D) \times (-8.35D, 8.35D)$, where D is the diameter of the cylinder and we take $D = 0.2$.
- A non-uniform grid 250×160 is adopted to discretize the computational domain, within which a uniform grid 60×60 is employed in the region $[-D, D] \times [-D, D]$.
- The small uniform mesh size is $h = 2D/60$ and time step length is $\Delta t = 0.4h \quad$ (*CFL* number is 0.4).



Example 2: flow past a stationary cylinder

- The drag coefficient C_d and the lift coefficient C_ℓ are respectively defined as

$$C_d = \frac{F_d}{U_\infty^2 D/2} \quad \text{and} \quad C_\ell = \frac{F_\ell}{U_\infty^2 D/2},$$

where the drag force F_d and the lift force F_ℓ are respectively calculated by

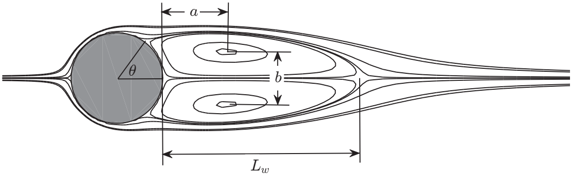
$$F_d = -\int_\Omega F_1 \, d\boldsymbol{x} \approx -\sum_{\boldsymbol{x}_{ij}} F_1 h^2 \quad \text{and} \quad F_\ell = -\int_\Omega F_2 \, d\boldsymbol{x} \approx -\sum_{\boldsymbol{x}_{ij}} F_2 h^2.$$

- The dimensionless vortex shedding frequency is called the Strouhal number and it is defined as $St = \frac{f_s}{U_\infty D}$.

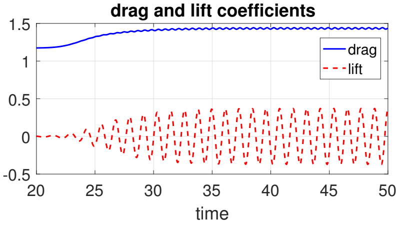
Example 2: numerical results at $Re = 40$

The comparison of experimental and numerical results of steady state wake dimensions and maximum drag coefficient for $Re = 40$

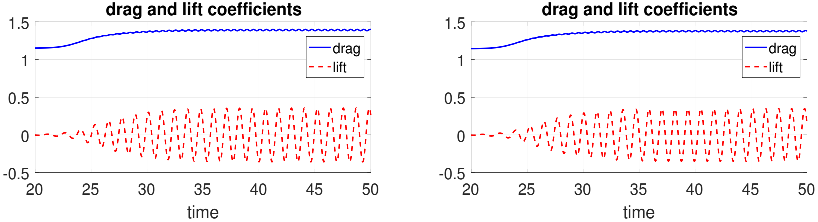
	$Re = 40$				
Methods	C_d	L_w/D	a/D	b/D	θ
Calhoun	1.62	2.18	–	–	54.2
Coutanceau-Bouard*	–	2.13	0.76	0.59	53.8
Linnick-Fasel	1.54	2.28	0.72	0.60	53.6
Su *et al.*	1.63	–	–	–	–
Taira-Colonius (B)	1.54	2.30	0.73	0.60	53.7
Tritton*	1.48	–	–	–	–
Ye *et al.*	1.52	2.27	–	–	–
Present method-P	1.59	2.20	0.71	0.60	51.2
Present method-PC	1.56	2.18	0.72	0.60	53.3



Example 2: drag and lift coefficients at $Re = 100$



present method-P



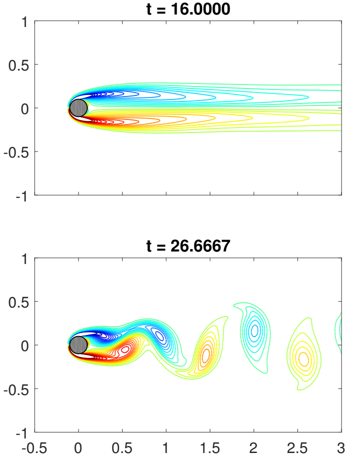
(left) present method-PC; (right) present method-PCC

Example 2: numerical results at $Re = 100$

The comparison of maximum drag and lift coefficients and Strouhal number for $Re = 100$

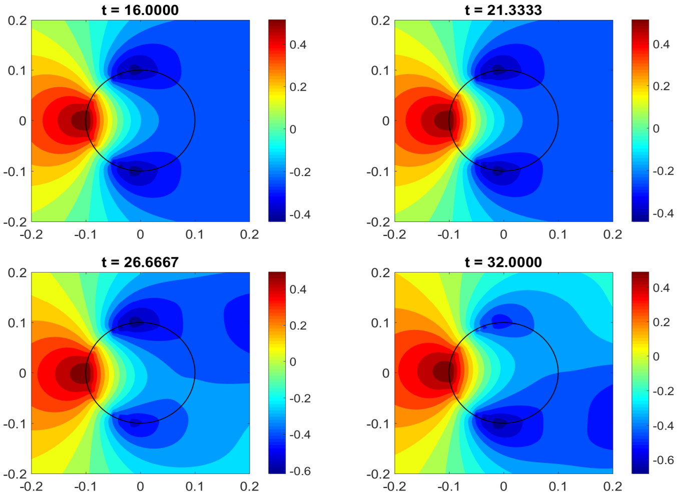
Methods	$Re = 100$		
	C_d	C_ℓ	St
Calhoun	1.36	0.30	0.175
Chiu *et at.*	1.36	0.30	0.167
Lai-Peskin	1.45	0.33	0.165
Liu *et al.*	1.36	0.34	0.164
Russell-Wang	1.39	0.32	0.170
Su *et al.*	1.40	0.34	0.168
Present method-P	1.43	0.37	0.171
Present method-PC	1.40	0.36	0.170

Instantaneous vorticity contours



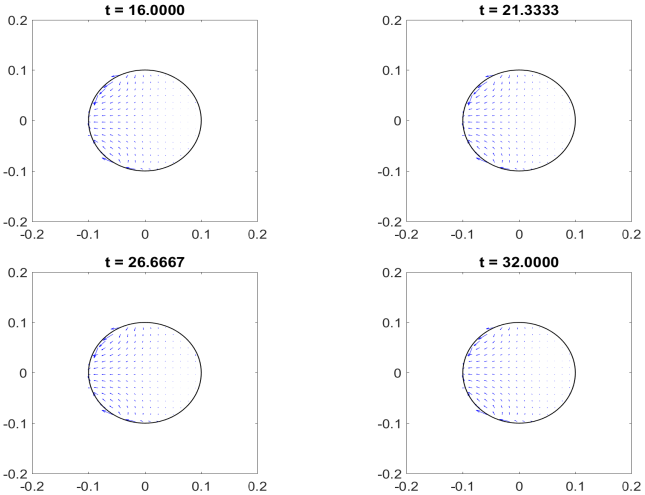
The direct-forcing IB projection method with PC based on the Choi-Moin scheme

Instantaneous pressure contours



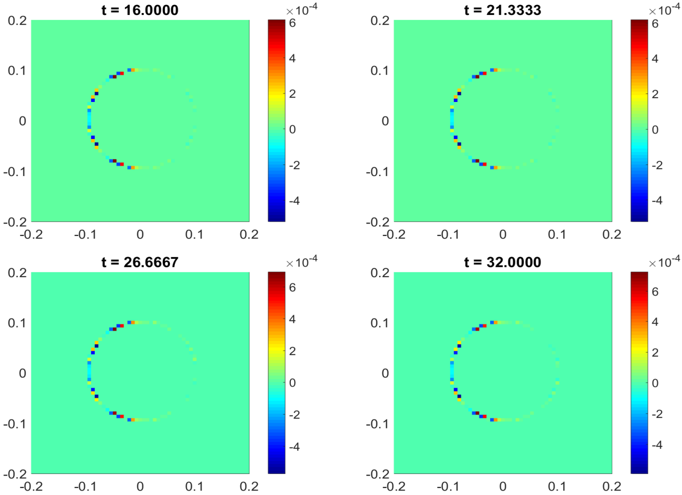
The direct-forcing IB projection method with PC based on the Choi-Moin scheme

Instantaneous virtual force F



The direct-forcing IB projection method with PC based on the Choi-Moin scheme

Instantaneous sink-source distribution

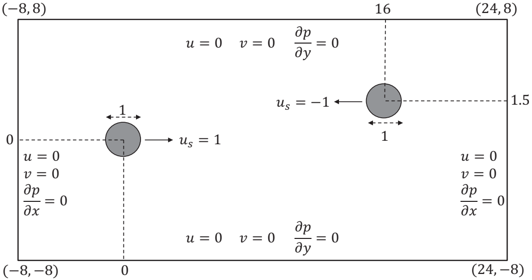


The direct-forcing IB projection method with PC based on the Choi-Moin scheme

Example 3: two cylinders moving towards each other

Problem setting–

- A uniform grid 640×320 is adopted to discretize the computational domain is $\Omega = (-8, 24) \times (-8, 8)$.
- $\Delta t = 1/200 \quad (CFL$ number is 0.1).
- The Reynolds number is $Re = 40$.



Example 3: two cylinders moving towards each other

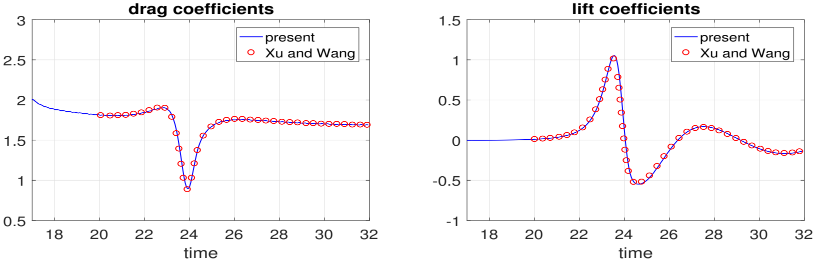
- The motion of the lower and upper cylinders are governed by setting the dynamics of their centers $(x_{\text{lc}}, y_{\text{lc}})$ and $(x_{\text{uc}}, y_{\text{uc}})$ to

$$x_{\text{lc}} = \begin{cases} \dfrac{4}{\pi}\sin\left(\dfrac{\pi t}{4}\right), & 0 \le t \le 16, \\ t - 16, & 16 \le t \le 32 \end{cases} \quad \text{and} \quad y_{\text{lc}} = 0,$$

and

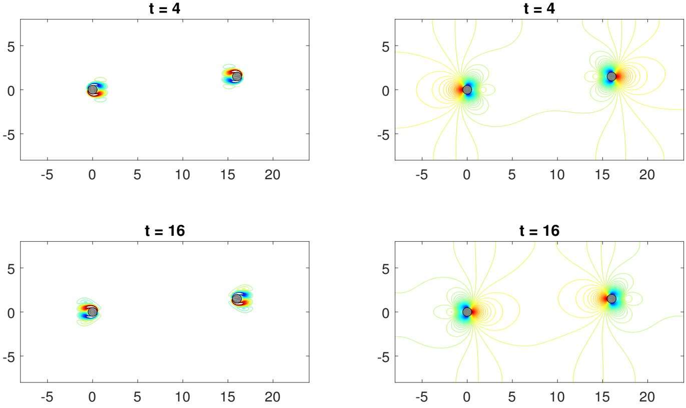
$$x_{\text{uc}} = \begin{cases} 16 - \dfrac{4}{\pi}\sin\left(\dfrac{\pi t}{4}\right), & 0 \le t \le 16, \\ 32 - t, & 16 \le t \le 32 \end{cases} \quad \text{and} \quad y_{\text{uc}} = 1.5.$$

Example 3: two cylinders moving towards each other



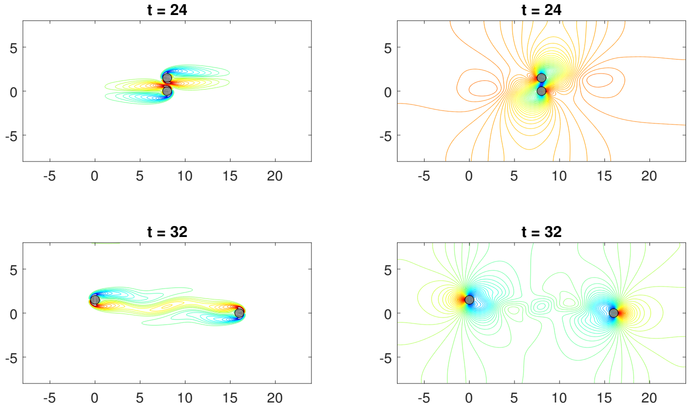
The time evolution of drag and lift coefficients, C_d and C_ℓ, for the upper cylinder in the flow around two cylinders compared with the results of Xu-Wang (JCP 2006)

Example 3: two cylinders moving towards each other



The flow around two cylinders moving towards each other for $Re = 40$ at different times: (left) contours of vorticity; (right) contours of pressure.

Example 3: two cylinders moving towards each other



The flow around two cylinders moving towards each other for $Re = 40$ at different times: (left) contours of vorticity; (right) contours of pressure.

Concluding remarks

1. We have developed a successful direct-forcing IB projection method with prediction-correction for simulating the fluid-solid interaction problems. This approach gives a significant improvement of the original direct-forcing IB projection method. proposed by Noor-Chern-Horng (CM 2009).
2. Further works are needed, including the improvements of the order of convergence, the extensions of the method to FSI problems without prescribed solid's velocity and the fluid-elastic body interaction problems.
3. Details of this approach can be found in

   T.-L. Horng, P.-W. Hsieh, S.-Y. Yang*, and C.-S. You, A simple direct-forcing immersed boundary projection method with prediction-correction for fluid-solid interaction problems, *submitted for publication*, 2017.

Governing equations of freely falling solid body

Consider a 2-D solid object of constant density ρ_s positioned at $\overline{\Omega}_s$ with centroid $\boldsymbol{X}_c$, translational velocity $\boldsymbol{u}_c$ and angular velocity ω. The velocity of the solid object is given by

$$\boldsymbol{u}_s(t,\boldsymbol{x}) = \boldsymbol{u}_c(t) + \omega(t) \times \boldsymbol{r}(t,\boldsymbol{x}), \quad \boldsymbol{r} := \boldsymbol{x} - \boldsymbol{X}_c, \quad \forall\, \boldsymbol{x} \in \overline{\Omega}_s(t).$$

From Newton's second law, we have

$$\begin{aligned} \frac{d\boldsymbol{u}_c}{dt} \int_{\Omega_s} \rho_f \, dV &= \int_{\partial\Omega_s} \boldsymbol{\tau} \cdot \boldsymbol{n} \, dS + \int_{\Omega_s} \rho_f \boldsymbol{F} \, dV + \int_{\Omega_s} \rho_f \boldsymbol{g} \, dV, \\ \boldsymbol{I}_f \frac{d\omega}{dt} &= \int_{\partial\Omega_s} \boldsymbol{r} \times (\boldsymbol{\tau} \cdot \boldsymbol{n}) \, dS + \int_{\Omega_s} \rho_f \boldsymbol{r} \times \boldsymbol{F} \, dV, \end{aligned}$$

where $\boldsymbol{\tau}$ is the stress tensor, $\boldsymbol{n}$ is the outward unit normal vector, ρ_f is the density of fluid, $\boldsymbol{g}$ is the gravity, $\boldsymbol{I}_f$ is the fluid replacement, and $\boldsymbol{F}$ is the virtual force, which is chosen to ensure $\boldsymbol{u} = \boldsymbol{u}_s$ on $\overline{\Omega}_s$.

From the viewpoint of solid body

The motion of solid object can also be described by translational and angular momentum of the solid body. Thus, we have

$$
\begin{aligned}
\frac{d\boldsymbol{u}_c}{dt}\int_{\Omega_s}\rho_s\,dV &= \int_{\partial\Omega_s}\boldsymbol{\tau}\cdot\boldsymbol{n}dS + \int_{\Omega_s}\rho_s\boldsymbol{g}\,dV,\\
\boldsymbol{I}_s\frac{d\omega}{dt} &= \int_{\partial\Omega_s}\boldsymbol{r}\times(\boldsymbol{\tau}\cdot\boldsymbol{n})\,dS,
\end{aligned}
$$

where $\boldsymbol{I}_s$ is the moment of inertia for the solid object. Since the virtual force $\boldsymbol{F}$ is chosen to make these two systems are equivalent, so we have the following equations of motion:

$$
\begin{aligned}
\frac{d\boldsymbol{u}_c}{dt}\int_{\Omega_s}\rho_s\,dV &= \int_{\Omega_s}(\rho_s-\rho_f)\boldsymbol{g}dV - \int_{\Omega_s}\rho_f\boldsymbol{F}\,dV + \frac{d\boldsymbol{u}_c}{dt}\int_{\Omega_s}\rho_f\,dV,\\
\boldsymbol{I}_s\frac{d\omega}{dt} &= -\int_{\Omega_s}\rho_f\boldsymbol{r}\times\boldsymbol{F}\,dV + \boldsymbol{I}_f\frac{d\omega}{dt}.
\end{aligned}
$$

Remarks on the equations of motion

1. The first term and second term in the right hand side of the first equation represent the difference of gravity and buoyant force and the drag, respectively.
2. The second equation can be further expressed as

$$M_s \frac{d\boldsymbol{u}_c}{dt} = (M_s - M_f)\boldsymbol{g} - \int_{\Omega_s} \rho_f \boldsymbol{F} dV + M_f \frac{d\boldsymbol{u}_c}{dt},$$

   where

$$M_s := \int_{\Omega_s} \rho_s \, dV = \int_{\Omega} \eta \rho_s \, dV, \quad M_f := \int_{\Omega_s} \rho_f \, dV = \int_{\Omega} \eta \rho_f \, dV,$$

   and

$$\eta(t, \boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \overline{\Omega}_s(t), \\ 0 & \boldsymbol{x} \notin \overline{\Omega}_s(t). \end{cases}$$

The two-way fluid-solid interaction problem

The fluid-solid interaction of the freely falling solid body with a virtual force can be formulated as the following initial-boundary value problem: *find* $\boldsymbol{u}$, p, $\boldsymbol{F}$, $\boldsymbol{u}_c$ *and* ω *such that*

$$\begin{gathered}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} + \boldsymbol{F} \quad t \in (0, T], \quad \boldsymbol{x} \in \Omega, \\
\nabla \cdot \boldsymbol{u} = 0 \quad t \in (0, T], \quad \boldsymbol{x} \in \Omega, \\
\boldsymbol{u} = \boldsymbol{u}_b \quad t \in (0, T], \quad \boldsymbol{x} \in \partial\Omega, \\
\boldsymbol{u} = \boldsymbol{u}_0 \quad t = 0, \quad \boldsymbol{x} \in \overline{\Omega}, \\
\boldsymbol{u} = \boldsymbol{u}_s := \boldsymbol{u}_c + \omega \boldsymbol{r} \quad \text{in } \overline{\Omega}_s, \\
M_s \frac{d\boldsymbol{u}_c}{dt} = (M_s - M_f)\boldsymbol{g} - \int_{\Omega_s} \rho_f \boldsymbol{F} dV + M_f \frac{d\boldsymbol{u}_c}{dt}, \quad \boldsymbol{u}_c(0) = \boldsymbol{u}_{c0}, \\
\boldsymbol{I}_s \frac{d\omega}{dt} = - \int_{\Omega_s} \rho_f \boldsymbol{r} \times \boldsymbol{F} dV + \boldsymbol{I}_f \frac{d\omega}{dt}, \quad \omega(0) = \omega_0.
\end{gathered}$$

Time-discretization of the equations of motion

Let $\boldsymbol{u}^n$, $p^{n-\frac{1}{2}}$, $\boldsymbol{F}^{n-\frac{1}{2}}$, $\boldsymbol{u}_c^n$, ω^n, $\boldsymbol{X}_c^n$ and $\boldsymbol{\theta}^n$ be given. At the time level t_{n+1}, we first predict the translational velocity and the angular velocity, denoted by $\boldsymbol{u}_c^{n+1,p}$ and $\omega^{n+1,p}$, by considering

$$\begin{aligned} M_s \frac{\boldsymbol{u}_c^{n+1,p} - \boldsymbol{u}_c^n}{\Delta t} &= (M_s - M_f)\boldsymbol{g} - \int_{\Omega_s} \rho_f \boldsymbol{F}^{n-\frac{1}{2}} dV + M_f \frac{\boldsymbol{u}_c^n - \boldsymbol{u}_c^{n-1}}{\Delta t}, \\ \boldsymbol{I}_s \frac{\omega^{n+1,p} - \omega^n}{\Delta t} &= -\int_{\Omega_s} \rho_f \boldsymbol{r}^n \times \boldsymbol{F}^{n-\frac{1}{2}} dV + \boldsymbol{I}_f \frac{\omega^n - \omega^{n-1}}{\Delta t}. \end{aligned}$$

Once $\boldsymbol{u}_c^{n+1,p}$ and $\omega^{n+1,p}$ are obtained, we compute the predicted solid center and rotational angle by taking

$$\frac{\boldsymbol{X}_c^{n+1,p} - \boldsymbol{X}_c^n}{\Delta t} = \frac{\boldsymbol{u}_c^n + \boldsymbol{u}_c^{n+1,p}}{2}, \quad \frac{\theta^{n+1,p} - \theta^n}{\Delta t} = \frac{\omega^n + \omega^{n+1,p}}{2},$$

update the solid domain Ω_s^{n+1}, and set the predicted solid velocity by

$$\boldsymbol{u}_s^{n+1,p} = \boldsymbol{u}_c^{n+1,p} + \omega^{n+1,p} \boldsymbol{r}^{n+1,p} \quad \text{with} \quad \boldsymbol{r}^{n+1,p} = \boldsymbol{X} - \boldsymbol{X}_c^{n+1,p}.$$

A two-stage direct-forcing IB projection method

- Based on the time-discretization of the equations of motion, we can devise a two-stage direct-forcing IB projection method for FSI problems without prescribed solid velocity.
- Please see some animations of the numerical simulations of freely falling solid bodies in the viscous, incompressible fluid.

References

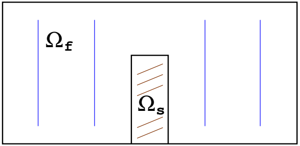
1. T. J. R. Hughes and J. E. Marsden, *A Short Course in Fluid Mechanics,* Publish or Perish, Inc., Berkeley, 1976.
2. A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics, 2nd Edition,* Springer-Verlag, New York, 1990.
3. H. C. Elman, D. J. Silvester, and A. J. Wathen, *Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics, 2nd Edition,* Oxford University Press, Oxford, 2014.
4. J. L. Guermond, P. Minev, and J. Shen, An overview of projection methods for incompressible flows, *Computer Methods in Applied Mechanics and Engineering,* 195 (2006), pp. 6011-6045.
5. T.-L. Horng, P.-W. Hsieh, S.-Y. Yang*, and C.-S. You, A simple direct-forcing immersed boundary projection method with prediction-correction for fluid-solid interaction problems, *submitted for publication,* 2017.

Fluid-elastic body interaction problems

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) containing an elastic body immersed in the incompressible viscous fluid flow for $t \in [0, T]$. The fluid domain is denoted by $\Omega_f(t)$ and the structure domain is denoted by $\Omega_s(t)$. That is, $\Omega_f(t) = \Omega \setminus \overline{\Omega}_s(t)$.

The governing equations of the fluid part of the fluid-elastic body interaction problem are given by

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times (0, T], \\
\nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times (0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_b \quad \text{on } \partial\Omega \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_s \quad \text{on } \partial\Omega_{sN} \times [0, T], \\
\boldsymbol{u} &= \boldsymbol{u}_0 \quad \text{in } (\Omega \setminus \overline{\Omega}_s) \times \{t = 0\}.
\end{aligned}
$$



$\boldsymbol{u}_s$: boundary velocity of the elastic body

Fluid-elastic body interaction problems

We assume that the elastic structure is quasi-static, isotropic and homogeneous which undergoes small deformations. Thus, we can use the linear elasticity to model the elastic body immersed in the fluid. The governing equations of the structure can be posed as follows:

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{v}_s) &= \boldsymbol{f}_s \quad \text{in } \Omega_s, \\ \boldsymbol{v}_s &= \mathbf{0} \quad \text{on } \partial\Omega_{sD}, \\ \boldsymbol{\sigma}_s(\boldsymbol{v}_s)\boldsymbol{n}_s &= -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \quad \text{on } \partial\Omega_{sN}, \end{cases}$$

- $\boldsymbol{v}_s$ is the displacement of the body; $\boldsymbol{f}_s$ is a given body force and we consider $\boldsymbol{f}_s = \mathbf{0}$ for simplicity; $\boldsymbol{\sigma}_s(\boldsymbol{v}_s)$ and $\boldsymbol{\sigma}(\boldsymbol{u}, p)$ are the stress tensors of the elastic body part and the fluid part, resp.
- $\boldsymbol{n}_s$ is the outward unit normal vector to $\partial\Omega_{sN}$ from the elastic body part and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega_{sN}$ from the fluid part. That is, $\boldsymbol{n}_s = -\boldsymbol{n}$ on $\partial\Omega_{sN}$.
- The boundary $\partial\Omega_{sD}$ of the elastic body is fixed on the global boundary $\partial\Omega$. This portion has no displacement for all time.

Linear elasticity

1. The strain tensor is defined as

$$\boldsymbol{\varepsilon}(\boldsymbol{w}) := \frac{1}{2}(\nabla \boldsymbol{w} + \nabla \boldsymbol{w}^{\top}), \quad \boldsymbol{w} \text{ could be } \boldsymbol{v}_s \text{ or } \boldsymbol{u}.$$

2. The Cauchy stress tensor of the elastic body is defined as

$$\boldsymbol{\sigma}_s(\boldsymbol{v}_s) := 2\mu_s \boldsymbol{\varepsilon}(\boldsymbol{v}_s) + \lambda_s(\nabla \cdot \boldsymbol{v}_s)I,$$

$$\mu_s := \frac{E}{2(1+\nu_s)} \quad \text{and} \quad \lambda_s := \frac{E\nu_s}{(1+\nu_s)(1-2\nu_s)}$$

are the Lamé parameters, E is the Young's modulus and ν_s with $0 < \nu_s < 1/2$ is the Poisson ratio. The upper limit of the Poisson ratio, $\nu_s \to (1/2)^-$, corresponds to an incompressible material.

3. The stress tensor of the fluid is defined as

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) := 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) - pI,$$

where μ is the dynamic viscosity of the fluid, which is defined as $\mu = \rho\nu$ and ρ is the fluid density and ν is the kinematic viscosity.

Strain and stress tensors

1. For 3-D case, the strain tensor can be expressed as

$$\varepsilon(\boldsymbol{w}) = \frac{\nabla \boldsymbol{w} + \nabla \boldsymbol{w}^\top}{2} = \frac{1}{2}\begin{bmatrix} 2w_{1x} & w_{1y}+w_{2x} & w_{1z}+w_{3x} \\ w_{1y}+w_{2x} & 2w_{2y} & w_{2z}+w_{3y} \\ w_{1z}+w_{3x} & w_{2z}+w_{3y} & 2w_{3z} \end{bmatrix}.$$

2. For 2-D case, the stress tensor of the elastic body is given by

$$\begin{aligned} \sigma_s(\boldsymbol{v}_s) &= 2\mu_s \varepsilon(\boldsymbol{v}_s) + \lambda_s(\nabla\cdot\boldsymbol{v}_s)I \\ &= \begin{bmatrix} 2\mu_s\frac{\partial v_{1s}}{\partial x} + \lambda_s\left(\frac{\partial v_{1s}}{\partial x} + \frac{\partial v_{2s}}{\partial y}\right) & \mu_s\left(\frac{\partial v_{1s}}{\partial y} + \frac{\partial v_{2s}}{\partial x}\right) \\ \mu_s\left(\frac{\partial v_{1s}}{\partial y} + \frac{\partial v_{2s}}{\partial x}\right) & 2\mu_s\frac{\partial v_{2s}}{\partial y} + \lambda_s\left(\frac{\partial v_{1s}}{\partial x} + \frac{\partial v_{2s}}{\partial y}\right) \end{bmatrix} \end{aligned}$$

and the stress tensor of the fluid is given by

$$\sigma(\boldsymbol{u}, p) = 2\mu\varepsilon(\boldsymbol{u}) - pI = \begin{bmatrix} 2\mu\frac{\partial u_1}{\partial x} - p & \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & 2\mu\frac{\partial u_2}{\partial y} - p \end{bmatrix}.$$

A direct-forcing IB method

Step 1: Predict the virtual force $\boldsymbol{F}^{n+1}$. Letting

$$\begin{aligned} \boldsymbol{F} &= \nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{v}_s) \\ &= \begin{bmatrix} (\lambda_s + 2\mu_s)\frac{\partial^2 v_{1s}}{\partial x^2} + \mu_s \frac{\partial^2 v_{1s}}{\partial y^2} + (\lambda_s + \mu_s)\frac{\partial^2 v_{2s}}{\partial x \partial y} \\ (\lambda_s + 2\mu_s)\frac{\partial^2 v_{2s}}{\partial y^2} + \mu_s \frac{\partial^2 v_{2s}}{\partial x^2} + (\lambda_s + \mu_s)\frac{\partial^2 v_{1s}}{\partial x \partial y} \end{bmatrix} \quad \text{in } \Omega_s(t) \end{aligned}$$

and taking time derivative, we have

$$\frac{d}{dt}\boldsymbol{F} = \frac{d}{dt}\Big(\nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{v}_s)\Big) = \nabla \cdot \boldsymbol{\sigma}_s\big(\frac{d}{dt}\boldsymbol{v}_s\big) = \nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{u}_s) \quad \text{in } \Omega_s(t).$$

Applying forward Euler scheme to discretize time variable, we obtain an approximation to the virtual force at time level $n+1$,

$$\boldsymbol{F}^{n+1} = \boldsymbol{F}^n + \Delta t \nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{u}_s^n) \quad \text{in } \Omega_s^n.$$

Step 2: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{F}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

A direct-forcing IB method (cont.)

Step 3: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{*}}{\Delta t}+\nabla p^{n+1} & = & \boldsymbol{0} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u}^{n+1} & = & 0 \quad \text{in } \Omega, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} & = & \boldsymbol{u}_b^{n+1}\cdot\boldsymbol{n} \quad \text{on } \partial\Omega. \end{cases}$$

Step 4: Find the displacement by letting

$$\boldsymbol{v}_s^{n+1}=\boldsymbol{v}_s^{n}+\Delta t\boldsymbol{u}_s^{n+1} \quad \text{in } \Omega_s^n,$$

and shift the elastic body to the new position by the displacement. Correct the virtual force by using the backward Euler scheme to

$$\frac{d}{dt}\boldsymbol{F}=\frac{d}{dt}\Big(\nabla\cdot\boldsymbol{\sigma}_s(\boldsymbol{v}_s)\Big)=\nabla\cdot\boldsymbol{\sigma}_s\Big(\frac{d}{dt}\boldsymbol{v}_s\Big)=\nabla\cdot\boldsymbol{\sigma}_s(\boldsymbol{u}_s) \quad \text{in } \Omega_s(t),$$

we obtain

$$\boldsymbol{F}^{n+1}:=\boldsymbol{F}^{n}+\Delta t\nabla\cdot\boldsymbol{\sigma}_s(\boldsymbol{u}_s^{n+1}) \quad \text{in } \Omega_s^{n+1}.$$

A two-stage direct-forcing IB method

First stage –

Step 1.1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 1.2: Determine $\boldsymbol{u}^{**}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{**} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{**} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{**} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Step 1.3: Predict the virtual force $\widetilde{\boldsymbol{F}}^{n+1}$ by setting

$$\widetilde{\boldsymbol{F}}^{n+1} = \boldsymbol{F}^n + \Delta t \nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{u}_s^{**}) \quad \text{in } \Omega_s^n.$$

A two-stage direct-forcing IB method (cont.)

Second stage –

Step 2.1: Solve for the intermediate velocity field $\boldsymbol{u}^*$,

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \nu \nabla^2 \boldsymbol{u}^* + (\boldsymbol{u}^n \cdot \nabla)\boldsymbol{u}^n = \widetilde{\boldsymbol{F}}^{n+1} & \text{in } \Omega, \\ \boldsymbol{u}^* = \boldsymbol{u}_b^{n+1} & \text{on } \partial\Omega. \end{cases}$$

Step 2.2: Determine $\boldsymbol{u}^{n+1}$ and p^{n+1} by solving

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^*}{\Delta t} + \nabla p^{n+1} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^{n+1} = 0 & \text{in } \Omega, \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \boldsymbol{u}_b^{n+1} \cdot \boldsymbol{n} & \text{on } \partial\Omega. \end{cases}$$

Step 2.3: Find the displacement

$$\boldsymbol{v}_s^{n+1} = \boldsymbol{v}_s^n + \Delta t \boldsymbol{u}_s^{n+1} \quad \text{in } \Omega_s^n,$$

and then shift the elastic body to the new position. Correct the virtual force $\widetilde{\boldsymbol{F}}^{n+1}$ as

$$\boldsymbol{F}^{n+1} := \boldsymbol{F}^n + \Delta t \nabla \cdot \boldsymbol{\sigma}_s(\boldsymbol{u}_s^{n+1}) \quad \text{in } \Omega_s^{n+1}.$$