Hyperbolic quenching problem with damping in the MEMS device.

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Join work with Jong-Shenq Guo of TKU
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Typically, the device of a microelectro mechanical system (MEMS) consists of an electric membrane hanged above a rigid ground plate, connected in series with a fixed voltage source and a fixed capacitor. As the potential difference applied between the membrane and the plate causes the membrane to be deflect.
We only derive the case of a membrane with constant dielectric permittivity profile.

**Coulomb Law:** The electrostatic force $F$ between two charges $q_1$ and $q_2$ placed at a distance $r$ apart is given in normalized units by

$$F = \frac{q_1 q_2}{r^2}.$$

If the two charges are uniformly distributed over two parallel plates subject to a capacitive influence of capacitance $C$ and a fixed electric voltage $V$, then we can write $q_1 = CV = q_2$ and $F$ becomes

$$F = \frac{C^2 V^2}{r^2}.$$

The electric potential $W$ can be expressed as

$$W = \frac{C^2 V^2}{r}.$$
Consider the case where the deformation of the membrane are small, so that each material point moves vertically over its reference position, and so that the material response is essentially linear. Assuming the deflection is upwards towards a ground plate at $z = 1$.

When the electric charges are not uniformly distributed as a result of a varying distance $1 - u(x)$, where $d = 1$ is the distance between the two plates in the absence of plate deformation, and $u(x)$ is the plate deformation variable, the electric potential becomes

$$E_W(u) = \int_{\Omega} \frac{C^2 V^2}{1 - u(x)} \, dx,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain.
Derivation of the model

On the other hand, the stretching energy in the elastic membrane is proportional to the changes in the area of the membrane from its unstretched configuration. Since we assume the membrane is held fixed at its boundary, we may then write

\[
\text{stretching energy} := T \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx,
\]

where \( T \) is the constant tension in the membrane. The linearized stretching energy is given by

\[
E_S(u) = \int_{\Omega} \frac{T}{2} |\nabla u|^2 \, dx.
\]

The total energy \( E = E_S + E_W \) may be represented as

\[
E(u) = \int_{\Omega} \left\{ \frac{T}{2} |\nabla u|^2 + \frac{C^2 V^2}{1 - u} \right\} \, dx,
\]

so that its Euler-Lagrange equation is

\[
T \Delta u + \frac{C^2 V^2}{(1 - u)^2} = 0 \text{ in } \Omega.
\]
For the dynamic deflection $u(x, t)$ of the membrane, let $\rho$ be the surface density of the membrane. By Newton’s 2nd Law and combining the superposition of the elastic and electrostatic forces, and a damping force $F_d$ which is linearly proportional to the velocity, that is,

$$F_d = -au_t.$$

We obtain the equation

$$\rho u_{tt} = -au_t + T\Delta u + \frac{C^2V^2}{(1-u)^2}.$$
Suppose the MEMS device have a capacitance $C$ depending on the displacement, connected in series with a capacitor of fixed capacitance $C_f$ and a source of fixed voltage $V_s$. By series circuit formula, the potential difference $V$ across the device will be

$$V = \frac{V_s}{1 + C/C_f}.$$ 

As a result, the capacitance $C$ can be evaluated as

$$C = C_0 \int_{\Omega} \frac{1}{1-u} \, dx,$$

for some $C_0$. The potential difference becomes

$$V = \frac{V_s}{1 + \alpha \int_{\Omega} \frac{1}{1-u} \, dx}.$$ 

Hence we obtain the equation

$$\rho u_{tt} = -au_t + T \Delta x + \frac{C^2 V_s^2}{(1-u)^2(1 + \alpha \int_{\Omega} \frac{1}{1-u} \, dx)^2}.$$
By proper rescaling, the model of dynamic deflection of an elastic membrane inside the MEMS device is as the following initial boundary value problem:

\[
\begin{aligned}
\varepsilon u_{tt} + u_t &= \Delta u + F(x, t, u), \quad &\text{in } \Omega \times (0, \infty) \\
\varepsilon u_t &= F(x, t, u) + \gamma u^{1/\alpha}, \quad &\text{on } \partial \Omega \times (0, \infty) \\
\varepsilon u(x, 0) &= u_0(x), \quad &\text{for } x \in \overline{\Omega} \\
\varepsilon u_t(x, 0) &= u_1(x), \quad &\text{for } x \in \overline{\Omega}
\end{aligned}
\]  

where \( \varepsilon > 0, \Omega \subset \mathbb{R}^N, 0 \leq u_0 < 1 \) on \( \overline{\Omega} \), \( u_0, u_1 \in C(\overline{\Omega}) \),

\[
F(x, t, u) := \frac{\lambda}{(1 - u)^2 (1 + \alpha \int_{\Omega} \frac{1}{1-u} \, dx)^2}
\]  

with \( \lambda > 0, \alpha \geq 0 \) on \( \overline{\Omega} \).
We say that \( u \) is a \textbf{weak solution} of (P1) on \( D_T := \Omega \times (0, T) \) for some \( T > 0 \), if

(i) \( u \in C^0(\bar{D}_T) \) such that the initial and boundary conditions in (P1) are satisfied,

(ii) \( |u| \leq 1 - \delta \) on \( \bar{D}_T \) for some \( \delta \in (0, 1) \),

(iii) the first order weak derivatives \( u_t, \nabla u \) of \( u \) are in \( L^2(D_T) \) such that

\[
\int_{\Omega} \psi(\varepsilon u_t + u) \, dx = \int_{0}^{t} \int_{\Omega} [\psi_t(\varepsilon u_t + u) - \nabla \psi \cdot \nabla u] \, dx \, dt + \int_{0}^{t} \int_{\Omega} \psi F(x, \tau, u) \, dx \, d\tau
\]

for all \( t \in (0, T) \) for any function \( \psi(x, t) \in C^1(\bar{D}_T) \) with \( \psi(0, t) = \psi(1, t) = 0 \) for all \( t \in [0, T) \).

We call a (weak) solution \textbf{quenches} (in finite time) if there is a \( T < \infty \) such that

\[
\lim \sup_{t \uparrow T} \max_{x \in \Omega} u(x, t) = 1.
\]
We say that \( u \) is a weak solution of (P1) on \( D_T := \Omega \times (0, T) \) for some \( T > 0 \), if

(i) \( u \in C^0(\bar{D}_T) \) such that the initial and boundary conditions in (P1) are satisfied,

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\[
\int_{\Omega} \psi(\varepsilon u_t + u) \, dx = \int_{0}^{t} \int_{\Omega} \left[ \psi(\varepsilon u_t + u) - \nabla \psi \cdot \nabla u \right] \, dx \, d\tau + \int_{0}^{t} \int_{\Omega} \psi F(x, \tau, u) \, dx \, d\tau \tag{1.2}
\]

for all \( t \in (0, T) \) for any function \( \psi(x, t) \in C^1(\bar{D}_T) \) with \( \psi(0, t) = \psi(1, t) = 0 \) for all \( t \in [0, T) \).

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Weak solutions

We say that $u$ is a **weak solution** of (P1) on $D_T := \Omega \times (0, T)$ for some $T > 0$, if

(i) $u \in C^0(\bar{D}_T)$ such that the initial and boundary conditions in (P1) are satisfied,

(ii) $|u| \leq 1 - \delta$ on $\bar{D}_T$ for some $\delta \in (0, 1)$,

(iii) the first order weak derivatives $u_t, \nabla u$ of $u$ are in $L^2(D_T)$ such that

$$
\int_{\Omega} \psi(\varepsilon u_t + u) \, dx = \int_{0}^{t} \int_{\Omega} \left[ \psi_{\tau}(\varepsilon u_{\tau} + u) - \nabla \psi \cdot \nabla u \right] \, dx \, d\tau + \int_{0}^{t} \int_{\Omega} \psi F(x, \tau, u) \, dx \, d\tau
$$

(1.2)

for all $t \in (0, T)$ for any function $\psi(x, t) \in C^1(\bar{D}_T)$ with $\psi(0, t) = \psi(1, t) = 0$ for all $t \in [0, T)$.

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Weak solutions

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(i) $u \in C^0(\bar{D}_T)$ such that the initial and boundary conditions in (P1) are satisfied,

(ii) $|u| \leq 1 - \delta$ on $\bar{D}_T$ for some $\delta \in (0, 1)$,

(iii) the first order weak derivatives $u_t, \nabla u$ of $u$ are in $L^2(D_T)$ such that

\[
\begin{align*}
\int_{\Omega} \psi(\varepsilon u_t + u) dx &= \int_0^t \int_{\Omega} [\psi_u(\varepsilon u_t + u) - \nabla \psi \cdot \nabla u] dx d\tau + \int_0^t \int_{\Omega} \psi F(x, \tau, u) dx d\tau \\
&= \int_0^t \int_{\Omega} \psi(\varepsilon u_t + u) dx d\tau
\end{align*}
\] (1.2)

for all $t \in (0, T)$ for any function $\psi(x, t) \in C^1(\bar{D}_T)$ with $\psi(0, t) = \psi(1, t) = 0$ for all $t \in [0, T)$.

We call a (weak) solution **quenches** (in finite time) if there is a $T < \infty$ such that

\[
\limsup_{t \uparrow T} \left\{ \max_{x \in \bar{\Omega}} u(x, t) \right\} = 1.
\]
1-D problem

Hereafter, we shall only discuss the case $N = 1$. Without loss of generality, by rescaling the variables $x$ and $t$, we may assume that $\varepsilon = 1$. By letting $\Omega = (0, 1)$, the first equation in (P1) becomes

$$u_{tt} + u_t = u_{xx} + \frac{\lambda h^2(u)}{[1 + \alpha I(u)]^2},$$

(1.3)

where

$$h(u) := \frac{1}{1 - u}, \quad I(u)(t) := \int_0^1 h(u(x, t)) \, dx.$$ 

Let $v = e^{t/2}u$. Then we can rewrite (1.3) as

$$v_{tt} = v_{xx} + \frac{1}{4} v + \frac{\lambda e^{t/2} h^2(e^{t/2}v)}{[1 + \alpha I(e^{t/2}v)]^2}.$$ 

(1.4)

The local existence can be obtained by contraction mapping principle.
Assuming $\alpha = 0$. Then (P1) becomes the following initial-boundary value problem:

$$
egin{aligned}
&u_{tt} + u_t = u_{xx} + \lambda h^2(u), & 0 < x < 1, \ t > 0 \\
&u(0, t) = u(1, t) = 0, & t > 0 \\
&u(x, 0) = u_0(x), & 0 \leq x \leq 1 \\
&u_t(x, 0) = u_1(x), & 0 \leq x \leq 1
\end{aligned}
$$

(2.1)

**Theorem 2.1**

Suppose that $u_0 \equiv 0$ and $u_1 \equiv 0$. If $\lambda \geq \pi^2/2$, then the solution of (2.1) quenches in finite time.
Introduction

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Quenching criterion for zero initial data with local source

**Sketch of the proof:**
We prove the theorem by the contrary. Suppose $0 \leq u < 1$. Define

$$G(t) := \int_0^1 u(x, t) \rho(x) \, dx, \quad \psi(x, t) := t \rho(x).$$

where

$$\rho(x) = \frac{\pi}{2} \sin(\pi x).$$

Then

$$0 \leq G(t) < 1, \quad \forall \ t \in [0, \infty) \quad \text{and} \quad G(0) = G'(0) = 0.$$ 

Multiplying the first equation in (2.1) by $\psi$ and integrating over $[0, 1] \times [0, t]$, the integration by parts gives

$$tG'(t) = -tG(t) + \int_0^t \left[ G'(\tau) + G(\tau) - \pi^2 \tau G(\tau) \right] d\tau + \lambda \int_0^t \tau \int_0^1 h^2(u) \rho(x) \, dx \, d\tau.$$ 

Differentiating the above equation with respect to $t$, we get

$$G''(t) = -G'(t) - \pi^2 G(t) + \lambda \int_0^1 h^2(u) \rho(x) \, dx.$$
Since $h^2$ is convex, it follows from Jensen’s inequality that

$$G''(t) + G'(t) \geq H(G(t)), \; \forall \; t \in [0, \infty).$$

where $H(z) := -\pi^2 z + \lambda h^2(z), \; -\infty < z < 1$. We can show that $G'(t) > 0$ forall $t > 0$. Then the increase of $H$ on $[0, 1]$ implies the following inequality

$$G''(t) + G'(t) \geq H(G(t)) \geq H(0) = \lambda > 0, \; \forall \; t \in [0, \infty). \quad (2.2)$$

Integrating (2.2) with respect to $t$ twice, and using the fact $G(t) < 1, \; \forall \; t$, we can deduce that

$$1 > G(t) > \frac{\lambda}{2} t^2 - t, \; \forall \; t \in [0, \infty),$$

which leads to a contradiction. Hence the theorem is proved.
Similarly, employing the standard convexity argument as in the proof of Theorem 2.1, we give the quenching criterion for nonzero initial data:

**Theorem 2.2**

Suppose that $G(0) \geq 0$ and $G'(0) \geq 0$ such that $G(0)^2 + G'(0)^2 \neq 0$. Then the solution of (2.1) must quench in finite time, provided that

\[
\begin{aligned}
\lambda &\geq \frac{\pi^2 [1 - G(0)]^3}{2}, & \text{if } 0 \leq G(0) < 1/3, \\
\lambda &> \pi^2 G(0) [1 - G(0)]^2, & \text{if } 1/3 \leq G(0) < 1.
\end{aligned}
\]  

(2.3)
Remark. For general bounded $\Omega \subset \mathbb{R}^N$ with $N > 1$, let $(\lambda^*, \rho)$ be the first eigen-pair of the problem

$$-\Delta \rho = \lambda^* \rho \quad \text{in } \Omega, \quad \rho = 0 \quad \text{on } \partial \Omega$$

such that $\int_{\Omega} \rho(x) dx = 1$. Define $\psi = t\rho$, then by a similar argument as the proof of Theorem 2.1 and 2.2, the solution of (P1) with $\varepsilon = 1$, quenches in finite time provided that

$$\lambda \geq \lambda^*/2 \quad \text{if } u_0 = u_1 = 0,$$

and

$$\begin{cases} 
\lambda \geq \frac{\lambda^*[1 - G(0)]^3}{2}, & 0 \leq G(0) < 1/3, \\
\lambda > \lambda^* G(0)[1 - G(0)]^2, & 1/3 \leq G(0) < 1,
\end{cases}$$

if $G(0) \geq 0$, $G'(0) \geq 0$, $G(0)^2 + G'(0)^2 \neq 0$. 
Criterion for global existence with local source

Multiplying the first equation of (2.1) by $u_t$ and integrate it over $[0, 1]$. Then we can define the total energy associated with the problem (P1) as

$$E(t) = \int_0^1 \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 - \lambda \Phi(u) \right] dx,$$

where $\Phi(z) = z/(1 - z)$. Moreover we have

$$\frac{dE}{dt} = - \int_0^1 u_t^2 dx \leq 0.$$

and hence $E(t) \leq E(0)$ for $t > 0$.

**Theorem 2.3**

Suppose that $u = u(x, t; \lambda)$ is the solution of (2.1) such that

$$0 < \lambda < \max_{0 \leq \delta \leq 1} \frac{\pi^2 \delta (1 - \delta)}{2(\pi \delta - \delta + 1)}.$$

If $E(0) \leq 0$, then there exists $\delta \in (0, 1)$ such that $|u(x, t; \lambda)| < 1 - \delta$ for all $x \in [0, 1]$ and $t \geq 0$. 
Criterion for global existence with local source

**Sketch of the proof:**
By assumption, there exists a $\delta > 0$ such that

$$\lambda < \frac{\pi^2 \delta (1 - \delta)}{2(\pi \delta - \delta + 1)} < \frac{\pi^2 \delta}{2}. \quad (2.5)$$

We claim that $|u(x, t; \lambda)| < 1 - \delta$ for all $x \in [0, 1]$ and $t \geq 0$. For the contradiction, suppose there exists $T < \infty$ such that

$$\max_{(x, t) \in [0, 1] \times [0, T]} u(x, t; \lambda) = 1 - \delta.$$

By Poincaré and Schwartz inequalities, we obtain

$$\pi^2 \int_0^1 u^2 \, dx \leq 2\lambda \left( \int_0^1 u^2 \, dx \right)^{1/2} \left[ 1 + \frac{1}{\delta} \left( \int_0^1 u^2 \, dx \right)^{1/2} \right],$$

and hence have the estimate for the $L^2$ norm of $u$, that is,

$$\|u\|_{L^2} \leq \frac{2\lambda}{\pi^2 - 2\lambda/\delta}. \quad (2.6)$$
Using the inequality

\[ 4u^2 \leq \int_0^1 u_x^2 dx, \]

and Poincaré and Schwartz inequalities again, we deduce that

\[ 4u^2 \leq 2\lambda \left( \int_0^1 u^2 dx \right)^{1/2} \left[ 1 + \frac{1}{\delta} \left( \int_0^1 u^2 dx \right)^{1/2} \right]. \]

Using (2.6), we get

\[ u^2 \leq \frac{\pi^2 \delta^2 \lambda^2}{(\pi^2 \delta - 2\lambda)^2}. \]

It follows from the assumption of \( \lambda \) (2.5) that

\[ u^2 < (1 - \delta)^2, \quad \forall (x, t) \in [0, 1] \times [0, T], \]

which contradicts the choice of \( T \). Hence the theorem is proved.
If $\alpha > 0$, then (P1) becomes the following initial-boundary value problem:

\[
\begin{aligned}
&u_{tt} + u_t = u_{xx} + \frac{\lambda h^2(u)}{[1 + \alpha l(u)]^2}, & 0 < x < 1, \ t > 0 \\
u(0, t) = u(1, t) = 0, & t > 0 \\
u(x, 0) = u_0(x), & 0 \leq x \leq 1 \\
u_t(x, 0) = u_1(x), & 0 \leq x \leq 1
\end{aligned}
\] (3.1)

For convenient, we denote

\[
\Psi(u)(t) = \frac{1}{1 + \alpha l(u)(t)}.
\]

We have the following observation

\[
\frac{d}{dt}\Psi(u)(t) = -\alpha \int_0^1 \frac{u_t h^2(u)}{[1 + \alpha l(u)]^2} \, dx.
\] (3.2)
To find the energy, we again multiply the first equation of (3.1) by $u_t$, integrate to over $[0, 1]$, and using (3.2), we can define the energy

$$E(t) = \int_0^1 \left[ \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right] dx + \frac{\lambda}{\alpha} \Psi(u)(t).$$  

and we also have $E(t) \leq E(0)$ for $t > 0$. By similarly argument in the proof of Theorem 2.3, we get

**Theorem 3.1**

The solution $u = u(x, t; \lambda)$ of (3.1) exists globally in time provided that

$$u_0 = u_1 = 0, \quad 0 < \lambda < 2\alpha(1 + \alpha),$$

or

$$\|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 < 4, \quad 0 < \lambda < \alpha \left( 2 - \frac{\|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2}{2} \right).$$
Quenching criterion for zero initial data with nonlocal source

If $u_0 = u_1 = 0$. Let $v = e^{t/2}u$, then $v$ satisfies

$$
\begin{align*}
\nu_{tt} - \nu_{xx} &= \frac{1}{4}v + \frac{\lambda e^{t/2} h^2(e^{-t/2} v)}{[1 + \alpha l(e^{-t/2} v)]^2}, & 0 < x < 1, \ t > 0, \\
v(0, t) &= v(1, t) = 0, & t > 0, \\
v(x, 0) &= \nu_t(x, 0) = 0, & 0 \leq x \leq 1.
\end{align*}
$$

(3.4)

Applying Picard iteration, we can construct a solution of (3.4) on the region $\mathcal{R} : [0, 1/2] \times [0, 1/2]$ such that $v(x, t)$ is independent of the spacial variable $x$ in some subregion of $\mathcal{R}$.

Lemma 3.2

There exists $t_0 \in (0, 1/2]$ such that the solution $v$ of (3.4) satisfies

$$
v(x, t) = V(t) = \max_{x \in [0, 1/2]} v(x, t)
$$

in the set $\{(x, t) \in \mathcal{R} \mid x \geq t, \ t < t_0\}$. 
Quenching criterion for zero initial data with nonlocal source

With this lemma, let $U(t) = e^{-t/2} V(t)$, then we can deduce that

$$U''(t) + U'(t) \geq \frac{\lambda}{(1 + \alpha)^2}, \quad U(0) = U'(0) = 0.$$ 

Hence we can give the following quenching criterion for the problem with zero initial data.

**Theorem 3.3**

Suppose that $\lambda \geq \hat{\lambda}(\alpha)$, where

$$\hat{\lambda}(\alpha) := \frac{(1 + \alpha)^2}{1/\sqrt{\bar{e}} - 1/2}.$$ 

Then the solution $u$ to (3.1) with $u_0 = u_1 = 0$ quenches in a finite time $T \leq 1/2$. 
Quenching criterion for nonzero initial data with nonlocal source

We set

$$A(t) = \int_0^1 u^2(x, t)dx.$$  

From (3.1) and integration by parts, and using the property of energy function, we obtain following inequality

$$A''(t) + A'(t) \geq Q(\lambda, \alpha),$$

where

$$Q(\lambda, \alpha) := \begin{cases}
-4E(0) + \frac{2\lambda}{\alpha}, & 0 < \alpha \leq 1/2, \\
-4E(0) + \frac{\lambda}{\alpha^2}, & \alpha > 1/2.
\end{cases}$$

Denote

$$\lambda^+(\alpha; u_0, u_1) := \begin{cases}
\frac{\alpha[\alpha I(u_0) + 1](\|u_0'\|_{L^2}^2 + \|u_1\|_{L^2}^2)}{\alpha I(u_0) - 1}, & 0 < \alpha \leq 1/2, \\
2\alpha^2[\alpha I(u_0) + 1](\|u_0'\|_{L^2}^2 + \|u_1\|_{L^2}^2) \left(1 - 4\alpha + \alpha I(u_0)\right), & \alpha > 1/2
\end{cases}.$$
Suppose $|u| < 1$ for $(x, t) \in [0, 1] \times [0, \infty)$. We deduced that

$$A(t) \geq \frac{1}{2} Q(\lambda, \alpha) t^2 + [A(0) + A'(0) - 1] t + A(0), \ \forall \ t \geq 0.$$ 

which leads to a contradiction. Therefore we have following theorem:

**Theorem 3.4**

Assume that the initial function $u_0$ satisfies

$$I(u_0) > \frac{1}{\alpha}, \text{ if } \alpha \in (0, 1/2]; \quad I(u_0) > 4 - \frac{1}{\alpha}, \text{ if } \alpha > 1/2.$$ 

Suppose that either $\lambda > \lambda^+ (\alpha; u_0, u_1)$, or

$$\lambda = \lambda^+ (\alpha; u_0, u_1) \quad \text{and} \quad \int_0^1 u_0^2(x) dx + 2 \int_0^1 u_0(x) u_1(x) dx > 1. \quad (3.5)$$

Then the solution of (3.1) quenches in finite time.
Remark. For general bounded $\Omega \subset \mathbb{R}^N$ with $N > 1$, let

$$\lambda^+(\alpha; u_0, u_1) := \begin{cases} \frac{\alpha [\alpha l(u_0) + 1] (\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2)}{\alpha l(u_0) - 1}, & 0 < \alpha \leq 1/(2|\Omega|), \\ \frac{2\alpha^2 |\Omega| [\alpha l(u_0) + 1] (\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2)}{1 - 4|\Omega| \alpha + \alpha l(u_0)}, & \alpha > 1/(2|\Omega|). \end{cases}$$

Under the assumption that the initial function $u_0$ satisfies

$$l(u_0) > 1/\alpha, \quad \text{if } 0 < \alpha \leq 1/(2|\Omega|); \quad l(u_0) > 4|\Omega| - 1/\alpha, \quad \text{if } \alpha > 1/(2|\Omega|),$$

if either $\lambda > \lambda^+(\alpha; u_0, u_1)$, or

$$\lambda = \lambda^+(\alpha; u_0, u_1) \quad \text{and} \quad \|u_0\|_{L^2}^2 + 2 \int_{\Omega} u_0(x) u_1(x) dx > 1,$$

then, by a similar argument as the proof of Theorem 3.4, the solution of (P1) with $\varepsilon = 1$, $f(x, t) \equiv 1$ quenches in finite time.
The existence of quenching curve

Consider the Cauchy problem:

\[
\begin{aligned}
(P2) & \quad \begin{cases}
    u_{tt} + u_t = u_{xx} + \lambda h^2(u) & \text{in } \mathbb{R} \times \{t > 0\}, \\
    u(x, 0) = u_0(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\
    u_t(x, 0) = u_1(x) & \text{on } \mathbb{R} \times \{t = 0\}.
\end{cases}
\end{aligned}
\]

To study the quenching curve for the problem (P2), we first prove the following lemma:

**Theorem 4.1**

Let $U(t)$ be the solution of

\[
\begin{aligned}
\begin{cases}
    U'' + U' = \lambda h^2(U), \\
    U(0) = U'(0) = 0.
\end{cases}
\end{aligned}
\]  

Then there exists $T_0 < \infty$ such that $U(t) \to 1$ as $t \to T_0$. 

Bo-Chih Huang

Hyperbolic quenching problem with damping in the MEMS device.
The existence of quenching curve

As before, we set \( v(x, t) = e^{t/2}u(x, t) \), then \( v \) satisfies

\[
\begin{cases}
  v_{tt} - v_{xx} = \frac{1}{4}v + \lambda e^{t/2}h^2(e^{-t/2}v) & \text{in } \mathbb{R} \times \{t > 0\}, \\
  v(x, 0) = u_0(x) =: v_0(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\
  v_t(x, 0) = \frac{1}{2}u_0(x) + u_1(x) =: v_1(x) & \text{on } \mathbb{R} \times \{t = 0\}.
\end{cases}
\]

and \( v \) can be expressed as

\[
v(x, t) = \bar{v}(x, t) + \frac{\lambda}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} W(v(y, \tau), \tau) dy \, d\tau,
\]

where

\[
\bar{v}(x, t) := \frac{1}{2}[v_0(x + t) + v_0(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} v_1(y) dy,
\]

\[
W(z, s) := \frac{1}{4\lambda} z + e^{s/2}h^2(e^{-s/2}z),
\]

as long as \( v(y, \tau) < e^{\tau/2} \) in the domain of dependence at \((x, t)\).
The existence of quenching curve

By showing that $v(y, \tau) > V(\tau)$, and thus $u(y, \tau) > U(\tau)$ in the domain of dependence at every $(x, t)$. By Lemma 4.1, we obtain following theorem:

**Theorem 4.2**

Suppose that $0 \leq u_0(x) < 1$ and $u_1(x) > 0$ in $\mathbb{R}$. Let $u$ be the solution of the Cauchy problem (P2). Then there exists a function $\phi$ defined in $\mathbb{R}$ with $0 < \phi(x) \leq T_0$ such that $u(x, t) < 1$ for $0 \leq t < \phi(x)$, $x \in \mathbb{R}$ and $u(x, t) \to 1$ as $t \uparrow \phi(x)$ for each $x \in \mathbb{R}$. 

The model for Spatial vary dielectric permittivity of the membrane

In previous case, the dielectric permittivity $\varepsilon$ of the membrane is constant. If we introducing a spatially varying dielectric permittivity $\varepsilon(x)$ of the membrane, the model will become

$$\varepsilon u_{tt} + u_t = \Delta u + \frac{\lambda f(x)}{(1 - u)^2 (1 + \alpha \int_{\Omega} \frac{1}{1-u} \, dx)^2}.$$  

The function $f(x)$ represents the varying dielectric properties of the membrane. Some physically suggested dielectric profiles are the power-law profile

$$f(x) = |x|^p, \quad p > 0.$$  

and the exponential-law profile

$$f(x) = e^{k(|x|^2-c)}, \quad k, c > 0.$$
Consider the Cauchy problem of (P1) with local source ($\alpha = 0$). By letting $v = 1 - u$, then $v$ satisfies
\[
\begin{align*}
\begin{cases}
v_{tt} + v_t &= \Delta v + v^{-2}, \\
v(x, 0) &= 1 - u_0(x), \\
v_t(x, 0) &= -u_1(x),
\end{cases}
\end{align*}
\tag{5.1}
\]
Suppose $v$ is the solution of (5.1) that quenches in finite time $T$, for each $a \in \mathbb{R}$, we employing the self-similar change of variables
\[
w_a(y, s) = (T - t)^{-2/3} v(x, t), \quad y = \frac{x-a}{T-t}, \quad s = -\log(T-t).
\]
Then $w$ satisfies the equation
\[
w_{ss} - \frac{1}{3} w_s + 2y \cdot \nabla w_s + \sum_{i,j} (y_i y_j - \delta_{ij}) w_{y_i y_j} + \frac{2}{3} y \cdot \nabla w = \frac{2}{9} w - \lambda w^{-2} - e^{-s} (w_s + y \cdot \nabla w - \frac{2}{3} w).
\]
We expect to find the lower and upper bound of $w$ in some functional space.
The fourth-order model for MEMS devices

When the elastic energy consists two parts: the stretching energy part and the bending energy part. Then the bending energy is given by

$$E_B(u) = \int_{\Omega} \frac{B}{2} (\Delta u)^2 \, dx,$$

for some constant $B > 0$. Thus the total energy $E = E_S + E_B + E_W$ may be represented as

$$E(u) = \int_{\Omega} \left\{ \frac{T}{2} |\nabla u|^2 + \frac{B}{2} (\Delta u)^2 + \frac{C^2 V^2}{1 - u} \right\} \, dx,$$

so that its Euler-Lagrange equation is

$$T \Delta u - B \Delta^2 u + \frac{C^2 V^2}{(1 - u)^2} = 0 \text{ in } \Omega.$$
This leads to the fourth-order model for MEMS devices

\[
\begin{cases}
    \varepsilon u_{tt} + u_t = \Delta u - \sigma \Delta^2 u + F(x, t, u), & \text{in } \Omega \times (0, \infty), \\
    u = 0, & \text{on } \partial\Omega \times (0, \infty), \\
    u(x, 0) = u_0(x), & \text{for } x \in \Omega, \\
    u_t(x, 0) = u_1(x), & \text{for } x \in \Omega,
\end{cases}
\]

where

\[
F(x, t, u) := \frac{\lambda}{(1 - u)^2 (1 + \alpha \int_{\Omega} \frac{1}{1-u} \, dx)^2}.
\]

1. The criterion of \(\lambda\) for existence of steady-state solution to (P3).
2. The quenching criterion for the solution to (P3).
3. The global existence criterion for the solution to (P3).
This leads to the fourth-order model for MEMS devices

\[
\begin{aligned}
\varepsilon u_{tt} + u_t &= \Delta u - \sigma \Delta^2 u + F(x, t, u), \quad \text{in } \Omega \times (0, \infty), \\
u &= 0, \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) &= u_0(x), \quad \text{for } x \in \overline{\Omega}, \\
u_t(x, 0) &= u_1(x), \quad \text{for } x \in \overline{\Omega},
\end{aligned}
\]

where

\[
F(x, t, u) := \frac{\lambda}{(1 - u)^2(1 + \alpha \int_{\Omega} \frac{1}{1-u} dx)^2}.
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u &= 0, \quad \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) &= u_0(x), \quad \text{for } x \in \bar{\Omega}, \\
u_t(x, 0) &= u_1(x), \quad \text{for } x \in \bar{\Omega},
\end{aligned}
\]

\[F(x, t, u) := \frac{\lambda}{(1 - u)^2 \left(1 + \alpha \int_{\Omega} \frac{1}{1 - u} \, dx\right)^2}.\]

- 1. The criterion of \(\lambda\) for existence of steady-state solution to (P3).
- 2. The quenching criterion for the solution to (P3).
- 3. The global existence criterion for the solution to (P3).
Thank you for your attention.