# Introduction to the course; sections 2.2-2.3

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# Linear Differential Equations

A differential equation is said to be **linear** if the following holds: If $y_1$ and $y_2$ are solutions, then $c_1y_1 + c_2y_2$ is also a solution, for every real numbers $c_1$ and $c_2$. This means two things:

- A multiple of a solution is also a solution
- The sum of two solutions is also a solution

**Example 1:** (a) Explain why the following equation is linear:

$$y'' - y' - 2y = 0$$

# Example 1, cont'd

Suppose $y_1$ and $y_2$ are solutions. What does this mean?

$$y_1'' - y_1' - 2y_1 = 0$$

and

$$y_2'' - y_2' - 2y_2 = 0$$

Multiply the first equation by $c_1$ and the second one by $c_2$. We obtain:

$$c_1 \left( y_1'' - y_1' - 2y_1 \right) = 0$$

and

$$c_2 \left( y_2'' - y_2' - 2y_2 \right) = 0$$

Note: $(c_1 y_1 + c_2 y_2)' = c_1 y_1' + c_2 y_2'$
$(c_1 y_1 + c_2 y_2)'' = c_1 y_1'' + c_2 y_2''$

# Example 1, cont'd

Add the two equations, and use the previous facts, to obtain:

$$(c_1 y_1 + c_2 y_2)'' - (c_1 y_1 + c_2 y_2)' - 2\,(c_1 y_1 + c_2 y_2) = 0$$

This means that $c_1 y_1 + c_2 y_2$ is also a solution.
**(b)** Suppose you know that $y_1 = e^{-x}$ and $y_2 = e^{2x}$ are solutions of

$$y'' - y' - 2y = 0.$$

Write down a general solution of this differential equation.

$$y = c_1\, e^{-x} + c_2\, e^{2x}$$

where $c_1$ and $c_2$ are constants to be determined from initial conditions.

# More Terminology

1. Linear first order DFQ:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

2. Linear first order DFQ with forcing term:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

The function $f(x)$ is called the forcing term (or forcing function). The equation in (1) is called the homogenous equation,while the equation in (2) is called non-homogeneous.

3. In general a **non-homogeneous DFQ** is **linear** if the **associated homogeneous equation** is linear:

Is the following DFQ linear?

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

**This is a linear second order differential equation with a forcing function.**

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$

Write is as:

$$f(y)\,dy = g(x)\,dx$$

Integrate both sides:

$$\int f(y)\,dy = \int g(x)\,dx$$

So, the general solution is given implicitly by:

$$F(y) = G(x) + C$$

**Example 2:** Solve:

$$\frac{dy}{dx} = \frac{x-5}{2y+3}$$

Write it as:

$$(2y+3)\ dy = (x-5)\ dx$$

# Example 2, cont'd

Integrate both sides:

$$y^2 + 3y = \frac{x^2}{2} - 5x + C$$

Solve for $y$: Complete the square:

$$\left(y + \frac{3}{2}\right)^2 = \frac{x^2}{2} - 5x + \frac{9}{4} + C$$

So

$$y = \frac{3}{2} \pm \sqrt{\frac{x^2}{2} - 5x + \frac{9}{4} + C}$$

# Solving linear first order, homogeneous

Solve:

$$a_1(x)y' + a_0(x)y = 0.$$

Re-write it in the form:

$$\frac{y'}{y} = \frac{-a_0(x)}{a_1(x)}$$

That is

$$\frac{dy}{y} = \frac{-a_0(x)}{a_1(x)}\,dx$$

So:

$$\ln|y| = \int \frac{-a_0(x)}{a_1(x)}\,dx + C$$

Or

$$y = C\,exp\left(\int \frac{-a_0(x)}{a_1(x)}\,dx\right).$$

# Solving linear first order, non-homogeneous

Solve:

$$a_1(x)y' + a_0(x)y = b(x).$$

Case 1: Suppose $a_0(x) = a_1'(x)$.The equation becomes:

$$a_1(x)y' + a_1'(x)y = b(x)$$

Or

$$(a_1(x)\,y)' = b(x)$$

That is

$$a_1(x)\,y = \int^x b(t)dt + C$$

or

$$y = \frac{1}{a_1(x)}\int b(x)dx + C$$

Case 2: $a_0(x) = 0$.The DFQ becomes:

$$a_1(x)y' = b(x).$$

So:

$$y' = \frac{b(x)}{a_1(x)}.$$

Or

$$y = \int \frac{b(x)}{a_1(x)} dx + C.$$

# Case 3: General Case

**First:** Put the DFQ in **standard form**:

$$y' + p(x)\, y = q(x)$$

**Second:** Find a function $\mu(x)$ (called a multiplier) such that the DFQ becomes:

$$(\mu(x)y)' = \mu(x)\, q(x)$$

Differentiating gives:

$$\mu(x)y' + \mu'(x)y = \mu(x)\, q(x)$$

Compare with:

$$\mu(x)\, y' + \mu(x)\, p(x)\, y = \mu(x)q(x)$$

We need

$$\mu'(x) = \mu(x)\, p(x)$$

or

$$\mu(x) = exp\left(\int p(x)\, dx\right)$$

The solution is then given by:

$$y(x) = \frac{1}{\mu(x)}\,\left(\int \mu(x)\, q(x)\, dx + C\right)$$

# Example 3

Solve

$$\frac{1}{x}y' - \frac{2}{x^2}\,y = x\,\cos x.$$

**Standard Form**:

$$y' - \frac{2}{x}\,y = x^2\,\cos x.$$

**Multiplier**: $\mu(x)$

$$\mu(x) = exp\left(\int p(x)\,dx\right) = exp\left(\int \frac{-2}{x}\,dx\right) = exp\,(-2\ln(x)) = x^{-2}$$

Multiply by $\mu(x)$ to get an exact derivative:

$$\left(x^{-2}\,y\right)' = \cos x.$$

Integrate both sides:

$$x^{-2}\,y = \sin x + C.$$

So: $y = x^2\,(\sin x + C)$

# Some issues to resolve re: notation, what it means to solve a differential equation, etc.

1. What do the following mean: $y'$, $y''$, etc?

The presumption is that the independent variable is $x$ and thus we are deriving with respect to $x$

2. What does it mean to "Solve the following DFQ..."?

Find the explicit function $y = y(x)$ which satisfies the given differential equation (or the given Initial Value Problem). It could also be an implicit solution of the form $F(x, y) = C$

3. What is this course?

In some sense it is a series of "cook book" recipes to solve various forms of differential equations.. with an added layer of modeling .. and intepretation.

So whenever possible, we will give a "tag" (or name) to each method as a sort of guide as to how one should proceed.

# What methods how we learned thus far?

- Separation of variables
- Multiplier (or Integrating Factor) Method

First, we review the "multiplier method" through a series of examples. You will help me "discover" this method through a series of examples, we will work out together on Elmo.

# Summary of "Multiplier Method":

The method is used to solve first order linear DFQ with a forcing term.

**Problem:** Solve

$$a_1(x)y' + a_0(x)\,y = b(x)$$

**Step 1: Standard Form.** Divide by $a_1(x)$ and change the equation into the form

$$y' + p(x)\,y = q(x)$$

**Step 2: Find the multiplier $\mu(x)$.**

$$\mu(x) = exp\left(\int p(x)\,dx\right)$$

**Step 3: Multiply the standard form by $\mu(x)$ and write the left hand side as a complete derivative and finish the problem**

# Example 1

This means that after multiplying, one gets:

$$(\mu(x)\, y)' = \mu(x)\, q(x).$$

Therefore:

$$\mu(x)\, y = \int \mu(x)\, q(x)\, dx + C$$

**Example 1:** Solve: $x\, y' + 2y = 5x^3$
**Step 1: Standard Form.** Divide by $x$:

$$y' + \frac{2}{x}\, y = 5\, x^2$$

**Step 2: Find the multiplier $\mu(x)$.**

$$\mu(x) = exp\left(\int \frac{2}{x}\, dx\right)$$

After cleaning (detail supplied on Elmo)

$$\mu(x) = x^2.$$

# Example 1, cont'd

**What is Step 3?**.. Multiply standard form by $\mu(x)$.
**Standard Form.**

$$y' + \frac{2}{x}\, y = 5\, x^2$$

multiply by $\mu(x) = x^2$:

$$x^2\, y' + 2x\, y = 5x^4$$

So

$$\left(x^2\, y\right)' = 5x^4$$

Integrate both sides:

$$x^2 y = x^5 + C$$

Simplifying leads to:

$$y = x^3 + \frac{C}{x^2}$$

# Example 1, second method: Variation of Parameters (Prob 30 in the text)

We are still solving: $x\,y' + 2y = 5x^3$

**Step 1: Find a solution $y_h$ to the homogeneous equation:**
$x\,y' + 2y = 0$ By separation of variables:

$$\frac{dy}{y} = -2\frac{dx}{x}$$

So: $\ln y = -2\ln x$, Or $y_h = x^{-2}$

**Step 2: Let $y = v(x)\,y_h$ and find the equation satisfied by $v$**
... After works $\frac{v'}{x} = 5x^3$
So $v' = 5x^4$ or $v = x^5 + C$. Therefore

$$y = v(x)\,y_h = \frac{v}{x^2} = \frac{x^5 + C}{x^2} = x^3 + \frac{C}{x^2}$$

# Example 2: Problem # 15, p. 55

Solve: $\left(x^2+1\right)\, y' + x\, y = x$.

**Step 1: Standard Form.** Divide by $\left(x^2+1\right)$:

$$y' + \frac{x}{x^2+1}\, y = \frac{x}{x^2+1}$$

**Step 2: Find the multiplier $\mu(x)$.**

$$\mu(x) = exp\left(\int \frac{x}{x^2+1}\, dx\right)$$

After cleaning (detail supplied on Elmo)

$$\mu(x) = \sqrt{x^2+1} = \left(x^2+1\right)^{1/2}.$$

# Example 2, cont'd

**Step 3:** Multiply standard form by $\mu(x) = \left(x^2+1\right)^{1/2}$ and finish the problem:

**Standard Form.**

$$y' + \frac{x}{x^2+1}\,y = \frac{x}{x^2+1}$$

After multiplication

$$\left(x^2+1\right)^{1/2}\,y' + x\left(x^2+1\right)^{-1/2}\,y = x\,\left(x^2+1\right)^{-1/2}$$

The magic:

$$\left(\left(x^2+1\right)^{1/2}\,y\right)' = x\,\left(x^2+1\right)^{-1/2}$$

Integrate both sides:

$$\left(x^2+1\right)^{1/2}\,y = \int x\,\left(x^2+1\right)^{-1/2}\,dx$$

# Example 2, cont'd

Note that

$$\int x\ (x^2+1)^{-1/2}\ dx = (x^2+1)^{1/2} + C$$

Therefore:

$$(x^2+1)^{1/2}\ y = (x^2+1)^{1/2} + C$$

Or

$$y = 1 + \frac{C}{(x^2+1)^{1/2}}$$

# Example 2, second method: Variation of Parameters (Prob 30 in the text)

We are still solving: $\left(x^2+1\right)\, y' + x\,y = x$.

**Step 1: Find a solution $y_h$ to the homogeneous equation:**
$\left(x^2+1\right)\, y' + x\,y = 0$
By separation of variables:

$$\frac{dy}{y} = -\frac{xdx}{x^2+1}$$

So: $\ln y = -\frac{1}{2}\ln(x^2+1)$, Or $y_h = (x^2+1)^{-1/2}$
**Step 2: Let $y = v(x)\, y_h$ and find the equation satisfied by $v$**
... After work: $(x^2+1)^{1/2}\, v = x$
So $v' = \frac{x}{(x^2+1)^{1/2}}$ or $v = (x^2+1)^{1/2} + C$. Therefore

$$y = v(x)\, y_h = v\,(x^2+1)^{-1/2} = 1 + C(x^2+1)^{-1/2},$$

Same solution as in the previous page.