單元 64: 三角函數的積分
(課本 §12.4)

根據微分與積分的互逆性, 由 §12.3 的 6 個微分公式可得對應的 6 個積分公式,

1. $\int \sin x dx = -\cos x + C$

2. $\int \cos x dx = \sin x + C$

3. $\int \sec^2 x dx = \tan x + C$

4. $\int \csc^2 x dx = -\cot x + C$

5. $\int \sec x \tan x dx = \sec x + C$

6. $\int \csc x \cot x dx = -\csc x + C$

<證> 1. 因爲

$$\frac{d}{dx}(\cos x) = -\sin x$$

同乘 (−1) 並根據微分的純量積規則, 得

$$\frac{d}{dx}(-\cos x) = \sin x$$

故由不定積分的定義, 即微分與積分互爲逆運算, 得

$$\int \sin x dx = -\cos x + C$$

2. 同理, 由

$$\frac{d}{dx}(\sin x) = \cos x$$

得

$$\int \cos x dx = \sin x + C$$

3. 因爲

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

故

$$\int \sec^2 x dx = \tan x + C$$

4. 由於

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

同乘 (-1), 得

$$\frac{d}{dx}(-\cot x) = \csc^2 x$$

且

$$\int \csc^2 x dx = -\cot x + C$$

5. 由

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

及不定積分的定義, 得

$$\int \sec x \tan x dx = \sec x + C$$

6. 因爲

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

故同乘 (-1), 得

$$\frac{d}{dx}(-\csc x) = \csc x \cot x$$

再根據微分與積分的互逆性, 得

$$\int \csc x \cot x dx = -\csc x + C$$

接著, 運用之前的代入法及分部積分, 可處理含三角函數的積分問題.

例 1. 試求 $\int \cos 3x dx$.

<解> 令 $u = 3x$, 得 $du = 3dx$, 故根據餘弦函數的積分公式,

$$\begin{aligned}\int \cos 3x dx &= \frac{1}{3}\int \cos 3x(3dx) = \frac{1}{3}\int \cos u du \\ &= \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C\end{aligned}$$

或因爲是線性轉換, 即若

$$\int f(x)dx = F(x) + C$$

則

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$$

故可直接根據餘弦函數的積分公式及積分的線性轉換公式, 求得

$$\int \cos 3x dx = \frac{1}{3}\sin 3x + C$$

例 2. 試求 $\int \sec(2x+1)\tan(2x+1)dx$.

<解> 根據積分的線性轉換公式,

$$\int \sec(2x+1)\tan(2x+1)dx = \frac{1}{2}\sec(2x+1)+C$$

例 3. 試求 $\int \frac{\sin x}{1+\cos x}dx$.

<解> 根據取

$$u = 1+\cos x, \quad du = -\sin x dx$$

的代入法並調整係數，得

$$\begin{aligned}\int \frac{\sin x}{1+\cos x}dx &= -\int \underbrace{\frac{1}{1+\cos x}}_{1/u}\underbrace{(-\sin x dx)}_{du} \\ &= -\ln|1+\cos x| + C\end{aligned}$$

例 4. 試求 $\int_0^{\pi/2} x\cos 2x dx$.

<解> 根據取

$$u = x, \quad dv = \cos 2x dx$$

以及

$$du = dx, \quad v = \frac{1}{2}\sin 2x$$

的分部積分, 得

$$\begin{aligned}
&\int_0^{\pi/2} x\cos 2x dx \\
= \ & \frac{1}{2}x\sin 2x\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin 2x dx \\
= \ & \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\pi/2} \\
= \ & \left(\frac{\pi}{4}\sin\pi + \frac{1}{4}\cos\pi\right) - \left(0 + \frac{1}{4}\cos 0\right) \\
= \ & \left(0 - \frac{1}{4}\right) - \frac{1}{4} = -\frac{1}{2}
\end{aligned}$$

例 5. 試求由 $t = 0$ 至 $t = \frac{\pi}{4}$ 曲線 $y = \sin 2t$ 下的區域面積.

<解> 由圖知, 區域面積

$$\begin{aligned}
A &= \int_0^{\pi/4}\sin 2t dt = -\frac{1}{2}\cos 2t\Big|_0^{\pi/4} \\
&= -\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos 0\right) \\
&= -\frac{1}{2}(0 - 1) = \frac{1}{2}
\end{aligned}$$

例 6. 設某股票在 t 週的收盤價為

$$f(t) = 30 + t\sin\frac{\pi}{6}t,\ 0 \le t \le 15$$

試求在 15 週內的平均週收盤價.

<解> 根據定義, 平均週收盤價

$$\begin{aligned} A &= \frac{1}{15-0}\int_0^{15}\left(30 + t\sin\frac{\pi}{6}t\right)dt \\ &= \frac{1}{15}(30t)\Big|_0^{15} + \frac{1}{15}\int_0^{15} t\sin\frac{\pi}{6}t dt \\ &= 30 + \frac{1}{15}\int_0^{15} t\sin\frac{\pi}{6}t dt \qquad (1) \end{aligned}$$

接著, 根據取

$$u = t, \quad dv = \sin\frac{\pi}{6}t dt$$

以及

$$du = dt, \quad v = -\frac{6}{\pi}\cos\frac{\pi}{6}t$$

的分部積分, (1) 式中, 積分式的不定積分

$$\begin{aligned} \int t\sin\frac{\pi}{6}t dt &= -\frac{6}{\pi}t\cos\frac{\pi}{6}t + \frac{6}{\pi}\int\cos\frac{\pi}{6}t dt \\ &= -\frac{6}{\pi}t\cos\frac{\pi}{6}t + \left(\frac{6}{\pi}\right)^2\sin\frac{\pi}{6}t + C \end{aligned}$$

故定積分

$$\int_0^{15} t\sin\frac{\pi}{6}t\,dt = -\frac{6}{\pi}t\cos\frac{\pi}{6}t + \left(\frac{6}{\pi}\right)^2\sin\frac{\pi}{6}t\Big|_0^{15}$$

$$= \left[-\frac{90}{\pi}\cos\frac{5\pi}{2} + \left(\frac{6}{\pi}\right)^2\sin\frac{5\pi}{2}\right] - \left[0 + \left(\frac{6}{\pi}\right)^2\sin 0\right]$$

$$= \left[-\frac{90}{\pi}(0) + \left(\frac{6}{\pi}\right)^2(1)\right] - (0+0) = \frac{36}{\pi^2}$$

最後, 代入 (1) 式, 得平均週收盤價

$$A = 30 + \left(\frac{1}{15}\right)\left(\frac{36}{\pi^2}\right) = 30 + \frac{36}{15\pi^2} \approx 30.24$$

即每股約 \$30.24.

另外 4 個基本三角函數的積分公式爲

7. $\int \tan x\,dx = -\ln|\cos x| + C$ 或

$$\int \tan x\,dx = \ln|\sec x| + C$$

8. $\int \cot x\,dx = \ln|\sin x| + C$

9. $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$

10. $\displaystyle\int \csc x dx = \ln|\csc x - \cot x| + C$

<證> 7. 改寫成正弦與餘弦的函數並根據取

$$u = \cos x, \quad du = -\sin x dx$$

的代入法以及對數律改寫, 得

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= -\int \frac{1}{\cos x}(-\sin x dx) \\
&= -\ln|\cos x| + C \\
&\overset{\text{或}}{=} \ln|\cos x|^{-1} + C \\
&= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C
\end{aligned}$$

8. 同理, 根據取

$$u = \sin x, \quad du = \cos x dx$$

的代入法, 得

$$\begin{aligned}
\int \cot dx &= \int \frac{\cos x}{\sin x} dx \\
&= \int \frac{1}{\sin x}(\cos x dx) = \ln|\sin x| + C
\end{aligned}$$

9. 同乘 $\sec x + \tan x$ 並根據取

$$u = \sec x + \tan x$$

以及

$$du = (\sec x \tan x + \sec^2 x)dx$$

的代入法，得

$$\begin{aligned}\int \sec x dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx \\ &= \int \frac{1}{\sec x + \tan x}(\sec^2 x + \sec x \tan x)dx \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

10. 同乘 $\csc x - \cot x$ 並根據取

$$u = \csc x - \cot x$$

以及

$$du = (-\csc x \cot x + \csc^2 x)dx$$

的代入法，得

$$\begin{aligned}\int \csc x dx &= \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} dx \\ &= \int \frac{1}{\csc x - \cot x}(\csc^2 x - \csc x \cot x)dx \\ &= \ln|\csc x - \cot x| + C\end{aligned}$$

Exercises

6. 根據積分的線性轉換公式以及三角函數的積分公式，

$$\int \csc^2 4x dx = -\frac{1}{4}\cot 4x + C$$

8. 根據取

$$u = x^2, \quad du = 2x dx$$

的代入法並調整係數，

$$\begin{aligned}
&\int x \sec x^2 \tan x^2 dx \\
&= \frac{1}{2}\int \sec x^2 \tan x^2 (2x dx) \\
&= \frac{1}{2}\sec x^2 + C
\end{aligned}$$

10. 根據積分的線性轉換公式，

$$\int \sec 2x \tan 2x dx = \frac{1}{2}\sec 2x + C$$

18. 根據積分的線性轉換公式，

$$\begin{aligned}
&\int \csc(1-x) dx \\
&= -\ln|\csc(1-x) - \cot(1-x)| + C
\end{aligned}$$

19. 根據積分的線性轉換公式以及正割函數的積分公式, 定積分

$$\begin{aligned}
&\int_0^{\pi/12} \sec 3x dx \\
&= \frac{1}{3}\ln|\sec 3x + \tan 3x|\Big|_0^{\pi/12} \\
&= \frac{1}{3}\left[\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| \right. \\
&\qquad\qquad \left. - \ln|\sec 0 + \tan 0|\right] \\
&= \frac{1}{3}[\ln(\sqrt{2}+1) - \ln(1+0)] \\
&= \frac{1}{3}\ln(\sqrt{2}+1)
\end{aligned}$$

25. 根據取

$$u = \tan x, \quad du = \sec^2 x dx$$

的代入法,

$$\begin{aligned}
\int \tan^3 x \sec^2 x dx &= \int u^3 du \\
&= \frac{1}{4}\tan^4 x + C
\end{aligned}$$

26. 根據取

$$u = \sec x, \quad du = \sec x \tan x dx$$

的代入法並改寫，得

$$\begin{aligned}\int \sec^7 x \tan x dx &= \int \sec^6 x \sec x \tan x dx \\ &= \int u^6 du = \frac{1}{7}\sec^7 x + C\end{aligned}$$

27. 根據取

$$u = \cot x - 1, \quad du = -\csc^2 x dx$$

的代入法，得

$$\begin{aligned}&\int \csc^2 x(\cot x - 1)^3 dx \\ &= -\int (\cot x - 1)^3(-\csc^2 x dx) \\ &= -\frac{1}{4}(\cot x - 1)^4 + C\end{aligned}$$

28. 根據取

$$u = 1 + \tan x, \quad du = \sec^2 x dx$$

的代入法，

$$\begin{aligned}&\int \sec^2 x\sqrt{1+\tan x} dx \\ &= \int \sqrt{1+\tan x}(\sec^2 x dx) \\ &= \frac{2}{3}(1+\tan x)^{3/2} + C\end{aligned}$$

31. 根據取

$$u = \sin(\ln x), \quad dv = dx$$

與

$$du = \cos(\ln x)\frac{1}{x}dx, \quad v = x$$

以及第二次

$$u = \cos(\ln x), \quad dv = dx$$

與

$$du = -\sin(\ln x)\frac{1}{x}dx, \quad v = x$$

的分部積分, 得

$$\begin{aligned}\int \sin(\ln x)dx &= x\sin(\ln x) - \int \cos(\ln x)dx \\ &= x\sin(\ln x) \\ &\qquad - \left[x\cos(\ln x) + \int \sin(\ln x)dx\right] \\ &= x\sin(\ln x) - x\cos(\ln x) \\ &\qquad - \int \sin(\ln x)dx\end{aligned}$$

其中最後一項的積分式剛好就是所求的原式.

故移項整理, 得

$$2\int \sin(\ln x)dx = x[\sin(\ln x) - \cos(\ln x)]$$

因此,

$$\int \sin(\ln x)dx = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$$

48. T, 因爲週期 2π, 故平移 2π 後不變; 或實際計算.

49. T, 首先,

$$\int_a^b \cos x dx = \sin x\Big|_a^b = \sin b - \sin a$$

又

$$\begin{aligned}\int_a^{b+2\pi} \cos x dx &= \sin x\Big|_a^{b+2\pi} \\ &= \sin(b+2\pi) - \sin a \\ &= \sin b - \sin a\end{aligned}$$

故相等.

50. T, 因爲

$$\begin{aligned}\sqrt[3]{-x}\sin(-2x) &= (-\sqrt[3]{x})(-\sin 2x) \\ &= \sqrt[3]{x}\sin 2x\end{aligned}$$

故被積函數爲偶函數, 對稱於 y-軸. 又

$$0 < x < \pi/2$$

時,

$$0 < 2x < \pi$$

且被積函數

$$\sqrt[3]{x}\sin 2x = (+)(+) > 0$$

爲正. 因此, 根據對稱性及取正値,

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \sqrt[3]{x}\sin 2x dx &= 2\int_0^{\pi/2} \sqrt[3]{x}\sin 2x dx \\ &> 0\end{aligned}$$

51. T, 因爲 $|\sin x|$ 爲偶函數, 故

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} |\sin x| dx &= 2\int_0^{\pi/2} \sin x dx \\ &= -2\cos x\Big|_0^{\pi/2} \\ &= -2(0-1) = 2\end{aligned}$$

同理, $|\cos x|$ 亦爲偶函數, 故

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} |\cos x| dx &= 2\int_0^{\pi/2} \cos x dx \\ &= 2\sin x\Big|_0^{\pi/2} \\ &= 2(1-0) = 2\end{aligned}$$

故相等.