Exercises for §5.1

1. Let \( f_n(x) = (x - 1/n)^2 \), \( 0 \leq x \leq 1 \). Does \( f_n \) converge uniformly?

**Sol.** We note that the pointwise limit of \( f_n(x) \) is \( x^2 \) since \( 1/n \to 0 \) as \( n \to \infty \).

Let \( f(x) = x^2 \), then

\[
\sup_{x \in [0,1]} \left| f_n(x) - f(x) \right| = \sup_{x \in [0,1]} \left| -2x/n + 1/n^2 \right| \leq \frac{2}{n} + \frac{1}{n^2} \to 0 \quad \text{as} \quad n \to \infty .
\]

which implies that the convergence is uniform.

3. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be uniformly continuous and let \( f_n \) converge uniformly to \( f \). Do you think that \( f \) must be uniformly continuous? Discuss.

**Sol.** Let \( \epsilon > 0 \) be given. Since \( f_n \to f \) uniformly, there exists \( N > 0 \) such that

\[
\sup_{x \in \mathbb{R}} \left| f_n(x) - f(x) \right| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \geq N.
\]

Since \( f_N \) is assumed to be uniformly continuous, there exists \( \delta > 0 \) such that

\[
\left| f_N(x) - f_N(y) \right| < \frac{\epsilon}{3} \quad \text{whenever} \quad |x - y| < \delta.
\]

Therefore,

\[
\left| f(x) - f(y) \right| \leq \left| f(x) - f_N(x) \right| + \left| f_N(x) - f_N(y) \right| + \left| f_N(y) - f(y) \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{whenever} \quad |x - y| < \delta.
\]

This implies that \( f \) is uniformly continuous on \( \mathbb{R} \).

Exercises for §5.2

1. Discuss the convergence and uniform convergence of

   a. \( f_n(x) = x^n/(n + x^n), \ x \geq 0, \ n = 1, 2, \cdots \)

   b. \( f_n(x) = e^{-x^2/n}, \ x \in \mathbb{R}, \ n = 1, 2, \cdots \)

**Sol.** We determine the pointwise limit first, and then see if the convergence is uniform.

a. For each \( x \in [0,1] \), \( f_n(x) \to 0 \), while for each \( x > 1 \), \( f_n(x) \to 1 \).

   Therefore, the pointwise limit of \( f_n(x) \) is \( f(x) = \begin{cases} 0 & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1. \end{cases} \)

   Moreover,

   \[
   \frac{d}{dx} f_n(x) = \frac{nx^{n-1}(n + x^n) - nx^{n-1}x^n}{(n + x^n)^2} = \frac{n^2x^{n-1}}{(n + x^n)^2} \geq 0
   \]
which implies that \( f_n \) is an increasing function. Therefore,
\[
\sup_{x \geq 0} |f_n(x) - f(x)| = \max \left\{ \sup_{x \in [0,1]} |f_n(x)|, \sup_{x > 1} |f_n(x) - 1| \right\}
= \max \{ |f_n(1)|, |f_n(0) - 1| \} = \frac{n}{n + 1} \not\to 0 \quad \text{as} \quad n \to \infty;
\]
thus the convergence is not uniform.

b. For each \( x \geq 0, f_n(x) \to 1 \) (why?). However, the convergence is not uniform since
\[
\sup_{x \in \mathbb{R}} |e^{-x^2/n} - 1| = 1 \not\to 0 \quad \text{as} \quad n \to \infty.
\]

4. Discuss the uniform convergence of \( \sum_{n=1}^{\infty} 1/(x^2 + n^2) \).

Sol. Let \( g_n(x) = 1/(x^2 + n^2) \) and \( M_n = \frac{1}{n^2} \). Then \( |g_n(x)| \leq M_n \) for all \( x \in \mathbb{R} \), and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by the \( p \)-series test or improper integral test. Therefore, the Weierstrass \( M \)-test implies that the series
\[
\sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}
\]
converges uniformly.

Exercises for Chapter 5

2. Determine which of the following sequences converge (pointwise or uniformly) as \( k \to \infty \). Check the continuity of the limit in each case.

a. \((\sin x)/k\) on \( \mathbb{R} \)

b. \(1/(kx + 1)\) on \([0,1]\)

c. \(x/(kx + 1)\) on \([0,1]\)

d. \(x/(1+kx^2)\) on \( \mathbb{R} \)

e. \((1, (\cos x)/k^2)\), a sequence of functions from \( \mathbb{R} \) to \( \mathbb{R}^2 \)

Sol.

a. The sequence of functions \((\sin x)/k\) on \( \mathbb{R} \) converges to 0 uniformly since
\[
\sup_{x \in \mathbb{R}} \left| \frac{\sin x}{k} - 0 \right| \leq \frac{1}{k} \to 0 \quad \text{as} \quad k \to \infty.
\]
b. The pointwise limit of the sequence $1/(kx + 1)$ on $]0, 1[$ is 0; however, the convergence is not uniform since

$$\sup_{x \in [0,1]} \left| \frac{1}{kx + 1} - 0 \right| \geq \frac{1}{k \cdot \frac{1}{k} + 1} = \frac{1}{2} \not\to 0 \quad \text{as} \quad k \to \infty.$$  

c. The pointwise limit of the sequence $x/(kx + 1)$ on $]0, 1[$ is 0 (why?). Moreover, the fact that

$$\frac{d}{dx} \frac{x}{kx + 1} = \frac{1}{(kx + 1)^2} > 0$$

implies that the function $\frac{x}{kx + 1}$ is increasing; thus

$$\sup_{x \in [0,1]} \left| \frac{x}{kx + 1} - 0 \right| \leq \frac{1}{k + 1} \to 0 \quad \text{as} \quad k \to \infty$$

which implies that the convergence is uniform.

d. The pointwise limit of the sequence $x/(1 + kx^2)$ on $\mathbb{R}$ is 0. As the previous case, we compute the derivative and find that

$$\frac{d}{dx} \frac{x}{1 + kx^2} = \frac{1 - kx^2}{(1 + kx^2)^2};$$

which implies that the maximum and the minimum of the function $x/(1 + kx^2)$ occurs at $x = -1/\sqrt{k}$ and $x = 1/\sqrt{k}$, respectively (why?). Therefore,

$$\sup_{x \in \mathbb{R}} \left| \frac{x}{1 + kx^2} - 0 \right| = \frac{1}{2\sqrt{k}} \to 0 \quad \text{as} \quad k \to \infty$$

which implies that the convergence is also uniform.

e. The pointwise limit of the sequence $(1, (\cos x)/k^2)$ is $(1, 0)$, and the convergence is uniform since

$$\sup_{x \in \mathbb{R}} \left| (1, \frac{\cos x}{k^2}) - (1, 0) \right| \leq \frac{1}{k^2} \quad \text{as} \quad k \to \infty.$$

3. Determine which of the following real series $\sum_{k=1}^{\infty} g_k$ converge (pointwise or uniformly). Check the continuity of the limit in each case.

a. $g_k(x) = \begin{cases} 0, & x \leq k \\ (-1)^k, & x > k. \end{cases}$

b. $g_k(x) = \begin{cases} 1/k^2, & |x| \leq k \\ 1/x^2, & |x| > k. \end{cases}$
c. \( g_k(x) = \left( \frac{(-1)^k}{\sqrt{k}} \right) \cos(kx) \) on \( \mathbb{R} \).

d. \( g_k(x) = x^k \) on \( ]0, 1[ \).

**Sol.** We first determine the sum of the series (if converges), and then determine the type of convergence and the continuity of the limit.

**a.** If \( x \leq 1 \) \( g_k(x) = 0 \) for all \( k \); thus the partial sum \( \sum_{k=1}^{n} g_k(x) \) vanishes. If \( x > 1 \), then \( g_k(x) \neq 0 \) only when \( k \leq [x] \) (where \([x]\) denotes the largest integer which is not greater than \( x \)); thus

\[
\sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{[x]} g_k(x). 
\]

Therefore,

\[
\sum_{k=1}^{\infty} g_k(x) = \begin{cases} 
0 & x \leq 1 \\
[x] \sum_{k=1}^{[x]} g_k(x) & x > 1.
\end{cases}
\]

To see if the convergence is uniform, we check the supremum of the difference between the partial sum and the limit, and find that

\[
\sup_{x \in \mathbb{R}} \left| \sum_{k=1}^{n} g_k(x) - \sum_{k=1}^{[x]} g_k(x) \right| = 1 \quad \text{if} \quad n = [x] + 2k - 1 \text{ for some } k \in \mathbb{N}.
\]

As a consequence, the convergence is not uniform.

**b.** Since \( g_k(x) \leq \frac{1}{k^2} \) and \( \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \), the Weierstrass \( M \)-test implies that the series \( \sum_{k=1}^{\infty} g_k(x) \) converges uniformly (thus the convergence is also pointwise). Moreover, it is clear that \( g_k \) is continuous on \( \mathbb{R} \); thus the partial sum \( \sum_{k=1}^{n} g_k \) is also continuous. By Proposition 5.1.4, the limit is continuous.

c. If \( x = \pi \), \( \cos(kx) = (-1)^k \); thus \( \sum_{k=1}^{\infty} \left( \frac{(-1)^k}{\sqrt{k}} \right) \cos(kx) \) diverges at \( x = \pi \).

Therefore, the series \( \sum_{k=1}^{\infty} g_k(x) \) does not converge pointwise.

d. This is Example 5.1.9 (c).

5. Suppose that \( f_k \to f \) uniformly, where \( f_k : A \subset \mathbb{R}^n \to \mathbb{R} \); \( g_k \to g \) uniformly, where \( g_k : A \to \mathbb{R}^m \); there is a constant \( M_1 \) such that \( \|g(x)\| \leq M_1 \) for all \( x \); and there is a constant \( M_2 \) such that \( |f(x)| \leq M_2 \) for all \( x \). Show
that \( f_k g_k \to fg \) uniformly. Find a counter example if \( M_1 \) or \( M_2 \) does not exist. Are \( M_1 \) and \( M_2 \) necessary for pointwise convergence?

**Sol.** Since \( f_k \to f \) uniformly, there exists \( N > 0 \) such that

\[
\sup_{x \in A} |f_k(x) - f(x)| < 1 \quad \text{whenever } k \geq N.
\]

Therefore,

\[
\sup_{x \in A} |f_k(x)| \leq \sup_{x \in A} |f(x)| + 1 \leq M_2 + 1 \quad \text{whenever } k \geq N.
\]

For \( k \geq N \),

\[
|f_k(x) g_k(x) - f(x) g(x)| \leq |f_k(x) - f(x)||g(x)| + |f_k(x)||g_k(x) - g(x)| \quad (0.1)
\]

\[
\leq M_1 |f_k(x) - f(x)| + (M_2 + 1)|g_k(x) - g(x)|;
\]

thus for \( k \geq N \)

\[
\sup_{x \in A} |f_k(x) g_k(x) - f(x) g(x)| \leq M_1 \sup_{x \in A} |f_k(x) - f(x)| + (M_2 + 1) \sup_{x \in A} |g_k(x) - g(x)|
\]

which converges to 0 as \( k \to \infty \) since \( f_k \to f \) and \( g_k \to g \) uniformly (why?). Therefore, \( f_k g_k \) converges uniformly to \( fg \).

If \( f \) and \( g \) both are not bounded functions, the limit on the right-hand side of (0.1) might not vanish. Let \( f_k(x) = g_k(x) = x + \frac{1}{k} \), \( f(x) = g(x) = x \), and \( A = \mathbb{R} \). Then \( f_k \to f \) and \( g_k \to g \) uniformly on \( A \) (why?), but

\[
|f_k(x)g_k(x) - f(x)g(x)| = (x + \frac{1}{k})^2 - x^2 = \frac{2x}{k} + \frac{1}{k^2};
\]

thus

\[
\sup_{x \in \mathbb{R}} |f_k(x) g_k(x) - f(x) g(x)| = \infty
\]

which implies that the convergence cannot be uniform.

For pointwise convergence, on the other hand, does not require that \( f \) and \( g \) are bounded. In fact, if \( f_k \to f \) and \( g_k \to g \) pointwise, \( f_k g_k \to fg \) pointwise.

---

**8. Does pointwise convergence of continuous functions on a compact set to a continuous limit imply uniform convergence on that set?**

**Sol.** Let \( K = [0, 1] \), and \( f_k : K \to \mathbb{R} \) be defined by

\[
f_k(x) = \begin{cases} 
  kx & 0 \leq x \leq \frac{1}{k} \\
  2 - kx & \frac{1}{k} < x \leq \frac{2}{k} \\
  0 & \text{otherwise.}
\end{cases}
\]
Then \( f_k \to 0 \) pointwise (why?), but the convergence is not uniform since\[
\sup_{x \in [0,1]} |f_k(x) - 0| = 1 \not\to 0 \quad \text{as} \quad k \to \infty.
\]

19. Prove that\[
\sum_{n=1}^{\infty} \left( \frac{\sin nx}{n^2} \right)^3
\]
defines a continuous function on all of \( \mathbb{R} \).

**Proof.** We only need to show that the series is continuous at each point \( a \in \mathbb{R} \).

To see this, let \( f_n(x) = \sum_{k=1}^{n} \left( \frac{\sin nx}{n^2} \right)^3 \) be the partial sum. We treat \( f_n \) as a sequence of functions defined on the interval \([-2|a|, 2|a|]\) and show that \( f_n \) converges uniformly. If the convergence is indeed uniform, then the limit \( f(x) = \sum_{n=1}^{\infty} \left( \frac{\sin nx}{n^2} \right)^3 \) must be continuous on \([-2|a|, 2|a|]\) by Proposition 5.1.4.

Nevertheless, on \([-2|a|, 2|a|]\) we find that \( \left| \frac{\sin nx}{n^2} \right|^3 \leq \frac{8|a|^3}{n^2} \), and it is clear that \( \sum_{n=1}^{\infty} \frac{8|a|^3}{n^2} < \infty \). Therefore, the Weierstrass M-test implies that the series \( \sum_{n=1}^{\infty} \left( \frac{\sin nx}{n^2} \right)^3 \) converges uniformly on \([-2|a|, 2|a|]\) (which is equivalent to that \( f_n \to f \) uniformly).

29. Discuss the uniform continuity of the following:

- **a.** \( f(x) = x^2, x \in [-1, 1[. \)
- **b.** \( f(x) = x^{1/3}, x \in [0, \infty[. \)
- **c.** \( f(x) = e^{-x}, x \in [0, \infty[. \)
- **d.** \( f(x) = x \sin(1/x), 0 < x \leq 1, f(0) = 0. \)
- **e.** \( f(x) = \sin[\ln(1 + x^3)], -1 < x \leq 1, f(-1) = 0. \)

**Sol.**

- **a.** We may extend the domain of \( f(x) \) by letting \( f(\pm 1) = 1 \). Then this extension, still denoted by \( f \), is continuous on \([-1, 1]\). Since \([-1, 1]\) is
compact, by Theorem 4.6.2, $f$ is uniformly continuous on $[-1, 1]$; thus $f$ is uniformly continuous on $] -1, 1]$. 

b. The function $f(x) = x^{1/3}$ is uniformly Hölder continuous on $x \in [0, \infty]$. In fact,

$$
\left| \frac{f(x) - f(y)}{|x - y|^{1/3}} \right| = \frac{|x - y|^{2/3}}{x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}} \leq 1 \quad \forall x, y > 0.
$$

Therefore, $|f(x) - f(y)| \leq |x - y|^{1/3}$ for all $x, y \geq 0$ (why?). This implies that $f$ is uniformly continuous on $[0, \infty)$.

c. The function $f(x) = e^{-x}$ is uniformly Lipschitz continuous on $[0, \infty]$ since the mean value theorem implies that

$$
|f(x) - f(y)| = |f'(\xi)||x - y| \leq |x - y|
$$

since $|f'| \leq 1$ on $[0, \infty]$. Therefore, for any given $\epsilon > 0$, $\delta = \epsilon$ will provide us the $\delta$ in the definition of uniform continuity. Therefore, $f$ is uniformly continuous on $[0, \infty]$.

d. The function $f : [0, 1] \to \mathbb{R}$ is continuous since by the squeeze theorem

$$
\lim_{x \to 0} f(x) = 0 = f(0)
$$

and it is obvious that $f$ is continuous at point $x \neq 0$. By Theorem 4.6.2, $f$ is uniformly continuous.

e. We note that $f$ is not continuous at $-1$ since the limit of $f$ as $x \to -1$ does not exist. This implies that $f$ cannot be uniformly continuous on $[-1, 1]$. \qed

33. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of increasing functions on $[0, 1]$, and suppose that $f_n \to 0$ pointwise. Must $f_n$ converge uniformly? What if $f_n$ just converges pointwise to some limit $f$?

**Sol.** The convergence is uniform since

$$
\sup_{x \in [0, 1]} |f_n(x) - 0| \leq |f_n(0)| + |f_n(1)| \to 0 \quad \text{as} \quad n \to \infty.
$$

Without the condition that $f_n$ is increasing or $f = 0$, then $f_n$ might not converges uniformly. For example,
1. The sequence $f_n$ in Exercise 8 converges pointwise to 0 but the convergence is not uniformly.

2. $f_n(x) = x^n$ which converges pointwise to $f = \begin{cases} 
0 & 0 \leq x < 1 \\
1 & x = 1 
\end{cases}$ but the convergence is not uniform.