Exercises for \S 6.1

4. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) and suppose there is a constant \( M \) such that for \( x \in \mathbb{R}^n \), \( \| f(x) \| \leq M \| x \| \). Prove that \( f \) is differentiable at \( x = 0 \) and that \( Df(0) = 0 \).

Proof:
\[
\| f(x) \| \leq M \| x \| \Rightarrow f(x) = 0
\]

Let \( Df(0) = 0 \)

\[
\lim_{x \to 0} \frac{\| f(x) - f(0) - Df(0)(x - 0) \|}{\| x - 0 \|} = \lim_{x \to 0} \frac{\| f(x) \|}{\| x \|} \leq \lim_{x \to 0} M \| x \| = 0
\]

By definition and \( 6.1.2 \) \( \Rightarrow Df(0) = 0 \).

5. If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and \( |f(x)| \leq |x| \), must \( Df(0) = 0 \).

Ans. No. Let \( f(x) = x \Rightarrow |f(x)| \leq |x| \) \( \Rightarrow Df(0) = 1 \)

Exercises for \( \S 6.4 \)

1. Use Theorem 6.4.1 to show that \( f(x, y) \) defined by

\[
f(x, y) = \frac{(xy)^2}{\sqrt{x^2 + y^2}} , (x, y) \neq (0, 0)
\]

and \( f(x, y) = 0 \) \( (x, y) = (0, 0) \)

is differentiable at \( (0, 0) \).

Proof:
\[
\frac{\partial f}{\partial x} = \frac{\frac{\partial}{\partial x} (xy)^2}{\sqrt{x^2 + y^2}} - \frac{1}{2} \frac{(xy)^2}{(x^2 + y^2)^{3/2}} \cdot 2y^2 = \frac{(xy)^2 - x(xy)^2}{x^2 + y^2} = \frac{x(xy)^2 - x(xy)^2}{x^2 + y^2}
\]

\[
= \frac{x^2 y^2 - x^2 y^2}{x^2 + y^2}
\]

\[
\Rightarrow \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x} = 0
\]

\[
\frac{\partial f}{\partial y} \text{ similar. By 6.4.1 } \Rightarrow f(x, y) \text{ is differentiable at } (0, 0)
\]
Investigate the differentiability of \( f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} \)

at \((0, 0)\) if \( f(0, 0) = 0 \)

Ans: If \( Df(x, y) \) exist

\[
\frac{\partial f(0, 0)}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = 0
\]

\[
\frac{\partial f(0, 0)}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = 0
\]

\[
\Rightarrow Df(0, 0) = (0, 0)
\]

\[
\Rightarrow Df(0, 0) (e_1, e_2) = 0 \quad \text{as} \quad e_2 \neq 0
\]

\[
\lim_{t \to 0} \frac{1}{t} f(te_1, te_2) = \lim_{t \to 0} \frac{te_1 e_2}{\sqrt{(te_1)^2 + (te_2)^2}} = 0 \quad \text{as} \quad e_2 \neq 0
\]

\( \Rightarrow f \) is not differentiable

5. Find a function \( f: \mathbb{R}^2 \to \mathbb{R} \) that is differentiable at each point but whose partials are not continuous at \((0, 0)\)

Ans: Let \( f(x, y) = \begin{cases} x^2 + y^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

\[
|f(x, y)| \leq 1|x|^2 \leq x^2y^2
\]

By exercises in §6.1 4

\( Df(0, 0) \) exists and \( Df(0, 0) = 0 \)

\[
\frac{\partial f(x, y)}{\partial x} = \begin{cases} 2x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}
\]

\[
\frac{\partial f(x, y)}{\partial y} = \begin{cases} 2y & \text{for } y \neq 0 \\ 0 & \text{for } x = 0 \end{cases}
\]

\( f \) is continuous when \( x \neq 0 \)

But \( \lim_{k \to 0} \frac{x \sin k - \cos k}{x} \) does not exist \( \Rightarrow \frac{\partial f(x, y)}{\partial x} \) is not continuous at \((0, 0)\)
Exercises for § 6.5

4. Write out the chain rule relating rectangular coordinates to spherical coordinates in three dimensions.

Ans: \( f(x, y, z) = f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \phi) \)

\[
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r}
\]

\[
\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \theta}
\]

\[
\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \phi} + 0
\]

5. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( F : \mathbb{R}^2 \to \mathbb{R} \) be differentiable and satisfy \( F(x, t) = 0 \) and \( \frac{\partial F}{\partial y} = 0 \).

Prove that \( f(x) = -\frac{\partial F}{\partial x} \) where \( y = f(x) \).

Pf:

\[
\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial y}
\]

\[
= \frac{\partial F}{\partial x} \frac{\partial x}{\partial y}
\]

\[
\Rightarrow f(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial x}{\partial y}}
\]

Exercises for § 6.6

1. Prove that

\[
\frac{d}{dt} \left( f(x_0 + t h) \right) \bigg|_{t=0} = Df(x_0) \cdot h
\]

by using the chain rule, where \( f : \mathbb{R}^n \to \mathbb{R}^n \).

Pf:

\[
Df \circ g : (x_0) = Df(g(x_0)) \circ Dg(x_0)
\]

Let \( x + th \) then

\[
\frac{d}{dt} f(x + t h) \bigg|_{t=0} = Df(x_0) \circ \frac{d}{dt} (x_0 + t h) = Df(x_0) \cdot h
\]
2. Prove the following (weak version of) L'Hôpital's rule: if \( f, g \) exist at \( x_0 \)
\[ g'(x_0) \neq 0 \] and \( f(x) = 0 \Rightarrow g(x) \), then
\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \]

Prove
\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f(x)}{g(x)} \]
\[ f(x_0), g(x_0) \text{ exist and } g'(x_0) \neq 0 \]
\[ = \frac{f(x_0)}{g(x_0)} \]

Let \( f: \mathbb{R}^n \to \mathbb{R}^m \) be differentiable with \( A \) convex and suppose \( \|g(x)\| \leq M \) for \( x \in A \). Prove \( \|f(x)-f(y)\| \leq M \|x-y\| \) for \( x, y \in A \). Do you think this is true if \( A \) is not convex?

By (6.7.1)
\[ f(x)-f(y)=\nabla f(c) \cdot (x-y) \]
\[ = \|f(x)-f(y)\| = \|\nabla f(c) \cdot (x-y)\| \]
\[ \leq M \|x-y\| \]
\[ \leq M \|x-y\| \]

(c) \( f \cdot x \): Let \( A = \mathbb{R} \setminus \{0\} \)
\[ f(x) = x^2 \]
\[ f(x) = 0 \quad x \in \mathbb{R} \]
\[ M = 0 \quad \forall \ x \in A \]
\[ \Rightarrow M \|x-y\| \leq M \|x-y\| \]
\[ x = y \leq x \leq y \]
Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable. Assume that for all \( x \in \mathbb{R} \), \( 0 \leq f(x) \leq f(x) \). Show that \( g(x) = e^{x}f(x) \) is decreasing. If \( f \) vanishes at some point, conclude that \( f \) is zero.

Prove (a) \( g'(x) = e^{x}f(x) + e^{x}f'(x) \)
\[ = e^{x}(f(x) - f(x)) \]
\[ > 0 \leq f(x) \leq f(x) \]
\[ \implies f(x) - f(x) \leq 0 \]
\[ \implies e^{x}(f(x) - f(x)) < 0 \]
\[ \implies g \text{ is decreasing} \]

Prove (b) let \( f \) vanishes at some point \( y \)
\[ f(y) = 0 \quad f(x) > 0 \quad x < y \]
\[ 0 > f(y) - f(x) = f(x)(y - x) > 0 \quad \forall x \]
\[ \implies f(x) = 0 \quad \forall x \in \mathbb{R} \]