Exercise for § 6.2

3. Let \( L \) be a linear map of \( \mathbb{R}^n \to \mathbb{R}^m \), let \( g : \mathbb{R}^n \to \mathbb{R}^m \) be such that \( \|g(x)\| \leq M \|x\|^2 \), and let \( f(x) = L(x) + g(x) \). Prove that \( Df(x) = L \).

**Pf:** By §6.1 exercise 4 \( \Rightarrow Dg(x) = 0 \).

By §6.2 example \( DL = L \).

\[ Df(x) = DL(x) + Dg(x) = L + 0 = L. \]

6. Discuss the possibility of defining \( Df \) for \( f \) a mapping from one normed space to another.

**Ans:** Let \( f : A \subseteq M \to N \), \( M, N \) are normed spaces.

\( f \) is said to be differentiable at \( x \in A \) if there is a linear function denoted \( Df(x) : M \to N \) and called the derivative of \( f \) at \( x_0 \), such that

\[ \lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df(x_0)(x - x_0) \|_N}{\| x - x_0 \|_M} = 0. \]

If \( M = \mathbb{R}^m, N = \mathbb{R}^m \)

\( Df(x_0) \) is continuous.

But if \( M, N \) are infinite dimensional space

\( Df(x_0) \) may not continue.
1. Let \( f(x) = x^2 \) if \( x \) is irrational and \( f(x) = 0 \) if \( x \) is rational. Is \( f \) continuous at 0? Is it differentiable at 0?

Proof:
Check if is continuous at 0

Let \( \varepsilon > 0 \) let \( \delta = \sqrt{\varepsilon} \)

\[ |f(x) - f(0)| = |x^2| < \delta \text{ if } |x| < \delta \]

Check if is differentiable at 0

\[ \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{x^2}{h} = 0 \text{ if } x \neq 0 \]

\[ \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \text{ if } x = 0 \]

\( \Rightarrow \ f \) is differentiable at 0

2. Is the local Lipschitz condition in Theorem 6.3.1 enough to guarantee differentiability?

Answer No.
Let \( f(x) = |x| \) \( x \in \mathbb{R} \)

\[ |f(x) - f(y)| = |x - y| \]

\( f \) is not differentiable at 0.
3 Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by
\[
f(x) = \begin{cases} 
x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]
and
\[
f(0) = 0
\]
Investigate the validity of Taylor's theorem for \( f \) about the point \( x = 0 \).

Ans:
\[
f''(x) = \lim_{x \to 0} \frac{f'(x) - f(x)}{x} = \lim_{x \to 0} \frac{x \sin \left( \frac{1}{x} \right)}{x} = 0 \quad (\because |\sin(\frac{1}{x})| \leq 1)
\]
\[
f'(x) = 2x \sin \left( \frac{1}{x} \right) - \cos \frac{1}{x}
\]
\[
\lim_{x \to 0} f'(x) \text{ does not exist} \quad \text{(\because \cos \frac{1}{x} \text{ oscillates between } +1 \text{ and } -1)}
\]
\[
\Rightarrow f''(x) \text{ is not continuous at } 0
\]

5 Compute the second-order Taylor formula for \( f(x,y) = e^{x \cos y} \) around \((0,0)\).

Ans:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= e^{x \cos y} & f(0,0) &= 1 \\
\frac{\partial f}{\partial y} &= -e^{x \cos y} & \frac{\partial^2 f}{\partial x \partial y} &= 0 \\
\frac{\partial^2 f}{\partial x^2} &= e^{x \cos y} & \frac{\partial^2 f}{\partial y^2} &= -e^{x \cos y} \\
\frac{\partial^2 f}{\partial x \partial y} &= -e^{x \cos y} & \frac{\partial^2 f}{\partial y \partial x} &= e^{x \cos y}
\end{align*}
\]
\[
f_{xx}(x,y) = 1 + (1,0) \cdot (h,k) + \left[ \begin{array}{c}
1 \\
0
\end{array} \right] \cdot \begin{pmatrix} h \\
k
\end{pmatrix} + R_2(h,k,0)
\]
\[
= 1 + h + \frac{1}{2} (h^2 - k^2) + R_2(h,k,0)
\]

where \( R_2(h,k,0) \to 0 \) as \((h,k) \to (0,0)\).
Exercise for 6.9

Prove that

\[
\begin{pmatrix}
  a & b \\
  b & d
\end{pmatrix}
\]

is negative definite iff \( a < 0 \) and \( ad - b^2 > 0 \)

1. Negative definite means,

\[
(x \cdot y) \begin{pmatrix}
  a & b \\
  b & d
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} > 0 \quad \text{if} \quad (x \cdot y) \neq (0, 0)
\]

2. \( ax^2 + 2bx + dy^2 < 0 \)

Let \( (x, y) = (1, 0) \) \( \Rightarrow a < 0 \)

Let \( y = 1 \), \( ax^2 + 2bx + d < 0 \) for all \( x \).

The function has maximum at \( ax + b = 0 \)

\( \Rightarrow x = -\frac{b}{a} \)

\( \Rightarrow a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + d < 0 \)

\( \frac{b^2}{a} - \frac{2b^2}{a} + d < 0 \)

\( ab - b^2 < 0 \)

"i" same way
4. (This exercise assumes knowledge of linear algebra) Let $A$ be a symmetric matrix. Show that $A$ is positive definite if and only if the eigenvalues of $A$ (which exist and are real, since $A$ is symmetric) are positive. Is this true if $A$ is not symmetric?

\[ A = U \Sigma U^T \]

\[ U^T U = U U^T = I \]

\[ \langle A x, x \rangle = \langle U \Sigma U^T x, x \rangle \]

\[ = \sum \lambda_i x_i^2 \]

\[ \geq 0 \text{ if } \lambda_1, \ldots, \lambda_n > 0 \text{ and } x \neq 0 \]

\( \Rightarrow \) $A$ is positive definite

(2) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

\[ \langle A u, u \rangle = \langle u, u \rangle = 1 \]

\[ \langle u, u \rangle = -1 \]

\[ \langle u, u \rangle = (1, -2) \]
Check that the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

has $\Delta \geq 0$ yet the matrix is not semidefinite.

Ans. Let $e_1 = (1, 0, 0)$, $e_3 = (0, 0, 1)$.

Then $\langle Ae_1, e_1 \rangle = 1$.

But $\langle Ae_3, e_3 \rangle = -1$ so $A$ is not semidefinite.