3. Let \( f(x) = x + 2x^2 \sin(x) \), \( x \neq 0 \), \( f(0) = 0 \). Show that \( f'(0) \neq 0 \) but that \( f \) is not locally invertible near 0. Why does this not contradict the inverse function theorem?

\[ \frac{df}{dx} = 1 + 4x \sin(x) + 2x^2 \cos(x) (-1) \frac{1}{2} \]
\[ = 1 + 4x \sin(x) - 2x^2 \frac{1}{2} \]
\[ \lim_{x \to 0} \frac{df}{dx} \text{ does not exist.} \]

\[ \frac{df(0)}{dx} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{1 + 2x \sin(\frac{x}{2})}{x} = 1 \]

\( \Rightarrow f' \) is not continuous at 0.

\( f \) is not locally invertible near 0 since it is not one-to-one in any neighborhood of 0.

4. Let \( L: \mathbb{R}^n \to \mathbb{R}^n \) be a linear isomorphism and \( f(x) = L(x) + g(x) \), where \( \|g(x)\| \leq M \|x\|^2 \) and \( f \) is \( C^1 \). Show \( f \) is locally invertible near 0.

\( \text{Pf:} \) By exercises for 6.6.2.

\[ df(0) = L \quad \Rightarrow L \text{ be a linear isomorphism} \]
\[ \det(L) \neq 0 \]

By inverse function theorem, \( f \) is locally invertible near 0.
3. In the system

\[ \begin{align*}
3x + 2y + z^2 + u + v^2 &= 0 \\
4x + 3y + z + u + v + w + z &= 0 \\
x + z + w + u^2 + z &= 0
\end{align*} \]

discuss the solvability for \( u, v, w \) in terms of \( x, y, z \) near \( x = y = z = 0 \) \( u = v = w = -2 \).

\textbf{Ans.} Let \( F_1 = 3x + 2y + z^2 + u + v^2 \)
\[ F_2 = 4x + 3y + z + u + v + w + z, \]
\[ F_3 = x + z + w + u^2 + z \]

\[ \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} \bigg|_{(0, 0, 0, 0, 0, -2)} = \begin{bmatrix}
\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\
\frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\
\frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ = \begin{bmatrix}
1 & 2v & 0 \\
0 & 1 & 1 \\
3v + 0 & 0 & 1
\end{bmatrix}
\]

\[ \det \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} = 1 \neq 0
\]

so \( u, v, w \) can be expressed in terms of \( x, y, z \) for \((x, y, z)\) in some neighborhood of \((0, 0, 0)\).
Discuss the solvability of

\[ y + x + uv = 0 \]
\[ uxy + v = 0 \]

for \( u, v \) in terms of \( x, y \), near \( x = y = u = v = 0 \), and check directly.

Ans. Let \( F_1 = y + x + uv \)

\[ F_2 = uxy + v \]

\[
\begin{bmatrix}
\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\
\frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}
\end{bmatrix}
\begin{bmatrix}
v \\
u
\end{bmatrix}
=
\begin{bmatrix}
v \\
u
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[ \det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \]

The implicit function theorem does not guarantee local solvability.