Problem 1. (10%) Figure 1 shows the region of the integration for the integral
\[ \int_0^1 \int_0^{1-x} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx. \]
Rewrite this integral as an equivalent iterated integral in the order of \( dxdzdy \).

**Sol:** The projection of the solid onto the \( yz \)-plane is the unit square \([0,1] \times [0,1]\). Moreover, the intersection of the plane \( y = 1 - x \) and the paraboloid \( z = 1 - x^2 \) is \( z = 1 - (1 - y)^2 = 2y - y^2 \) which divides the unit square into two pieces. Let \( A \) be the piece adjacent to the \( z \)-axis and \( B \) be the piece adjacent to the \( y \)-axis.

In other words,

\[
A \equiv \left\{ (y, z) \mid 0 \leq y \leq 1, 2y - y^2 \leq z \leq 1 \right\},
\]
\[
B \equiv \left\{ (y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq 2y - y^2 \right\}.
\]

Therefore, the integral can be also written as
\[
\int\int_A \int_0^{\sqrt{1-z}} f(x, y, z) \, dz \, dx + \int\int_B \int_0^{1-y} f(x, y, z) \, dx \, dz \, dy
\]
\[
= \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x, y, z) \, dz \, dx \, dy + \int_0^1 \int_0^{2y-y^2} \int_0^{\sqrt{1-z}} f(x, y, z) \, dz \, dx \, dy.
\]
Problem 2. (15%) Evaluate the triple integral
\[
\iiint_T xyz \, dV,
\]
where T is the solid tetrahedron with vertices (0, 0, 0), (1, 0, 0), (1, 1, 0) and (1, 0, 1).

Sol: First of all, the plane passing through (0, 0, 0), (1, 1, 0) and (1, 0, 1) is \(x - y - z = 0\) or \(z = x - y\).

Therefore, the tetrahedron T can be expressed as
\[
T = \left\{ (x, y, z) \mid (x, y) \in D, \ 0 \leq z \leq x - y \right\},
\]
where \(D\) is the triangular region on \(xy\)-plane with vertices (0, 0), (1, 0) and (1, 1). As a consequence,
\[
\iiint_T xyz \, dV = \int_0^1 \int_0^x \int_0^{x-y} xyz \, dz \, dy \, dx = \int_0^1 \int_0^x \frac{xy(x-y)^2}{2} \, dy \, dx
\]
\[
= \int_0^1 \int_0^x \left[ -\frac{x(x-y)^3}{2} + \frac{x^2(x-y)^2}{2} \right] \, dy \, dx
\]
\[
= \int_0^1 \left[ \frac{x(x-y)^4}{8} - \frac{x^2(x-y)^3}{6} \right] \bigg|_{y=x}^{y=0} \, dx
\]
\[
= \int_0^1 \left[ -\frac{x^5}{8} + \frac{x^5}{6} \right] \, dx = \frac{x^6}{144} \bigg|_{x=1}^{x=0} = \frac{1}{144}. \quad \square
\]

Problem 3. Find the volume of the solid that lies between the paraboloid \(z = x^2 + y^2\) and the sphere \(x^2 + y^2 + z^2 = 2\) using

1. (15%) the cylindrical coordinate, and
2. (15%) the spherical coordinate.

Sol: First we find the intersection of the paraboloid and the sphere. If \((x, y, z)\) is on the intersection, then \(z + z^2 = 2\) which implies \(z = 1\) or \(z = -2\) which is impossible. Therefore, the paraboloid and the sphere intersection at \(x^2 + y^2 = 1\).
1. Using the cylindrical coordinates, the paraboloid can be expressed as $z = r^2$ and the upper half sphere can be expressed as $z = \sqrt{2 - r^2}$. Therefore, the required volume is

$$
\int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2 - r^2}} rdz dr d\theta = \int_0^{2\pi} \int_0^1 \left[ r\sqrt{2 - r^2} - r^3 \right] dr d\theta = 2\pi \left[ -\frac{1}{3}(2 - r^2)^{\frac{3}{2}} - \frac{1}{4}r^4 \right]_{r=0}^{r=1}
$$

$$
= 2\pi \left[ \left( -\frac{1}{3} - \frac{1}{4} \right) - \left( -\frac{2\sqrt{2}}{3} \right) \right] = \frac{2\pi}{3} \left( 2\sqrt{2} - \frac{7}{4} \right).
$$

2. Using the spherical coordinates, the paraboloid can be expressed as $\rho = \cos \varphi \sin^2 \varphi$, and the sphere can be expressed as $\rho = \sqrt{2}$. Therefore, the required volume is

$$
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{2\pi}^{\pi} \int_0^\rho \rho^2 \sin \varphi d\rho d\theta d\varphi + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \int_{2\pi}^{\pi} \int_0^{\cos \varphi} \rho^2 \sin \varphi d\rho d\theta d\varphi 
$$

$$
= \frac{2\sqrt{2}}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \varphi d\varphi d\theta + \frac{2\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos \varphi} \cos ^3 \varphi \sin \varphi d\rho d\theta d\varphi 
$$

$$
= \frac{2\sqrt{2}}{3} \times 2\pi \times \left[ -\cos \varphi \right]_{\varphi=0}^{\frac{\pi}{2}} + \frac{2\pi}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \sin^2 \varphi) \cos \varphi \sin^5 \varphi d\varphi 
$$

$$
= \frac{2\pi}{3} \left( 2\sqrt{2} - 2 \right) + \frac{2\pi}{3} \int_1^1 \frac{1 - u^2}{u^5} du \quad \text{(by letting } u = \sin \varphi \text{)}
$$

$$
= \frac{2\pi}{3} \left( 2\sqrt{2} - 2 \right) + \frac{2\pi}{3} \left[ -\frac{1}{4}u^{-4} + \frac{1}{2}u^{-2} \right]_{u=1}^{u=\frac{\sqrt{3}}{2}}
$$

$$
= \frac{2\pi}{3} \left( 2\sqrt{2} - 2 \right) + \frac{2\pi}{3} \left[ \left( -\frac{1}{4} + \frac{1}{2} \right) - \left( -\frac{4}{4} + \frac{2}{2} \right) \right]
$$

$$
= \frac{2\pi}{3} \left( 2\sqrt{2} - 2 + \frac{1}{4} \right) = \frac{2\pi}{3} \left( 2\sqrt{2} - \frac{7}{4} \right).
$$

where we note that the first integral is the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere, while the second integral is the volume of the solid below the cone and above the paraboloid.

Problem 4. Let $R$ be the region bounded by $y = 3x$, $y = \sqrt{3}$ and the hyperbola $xy = 3$. Find the double integral $\iiint_R xy \, dA$ by

1. (10%) Plot the region $S$ on the $uv$-plane which corresponds $R$ on the $xy$-plane.

2. (15%) Use the change of variables $x = \frac{u}{v}$ and $y = v$ and the change of variable formula to compute the double integral.

Sol:

1. Since $x = \frac{u}{v}$ and $y = v$, the curve on the $uv$-plane corresponding to $y = 3x$ is

$$
v = \frac{3u}{v} \quad \text{or} \quad u = \frac{v^2}{3},
$$
while the hyperbola $xy = 3$ on the $xy$-plane corresponds to $u = 3$ on the $uv$-plane. Moreover, the curve corresponding to $y = \sqrt{3}$ on the $xy$-plane is $v = \sqrt{3}$ on $uv$-plane. Therefore,

$$\begin{array}{c}
\dfrac{\partial (x, y)}{\partial (u, v)} = \left| \begin{array}{cc}
x_u & x_v \\
y_u & y_v
\end{array} \right| = \left| \begin{array}{cc}
\frac{1}{v} & -\frac{u}{v^2} \\
0 & 1
\end{array} \right| = \frac{1}{v}.
\end{array}$$

Therefore,

$$\int_R xydA = \int_S \int \frac{u}{|v|} dvdu = \int_1^{\sqrt{3} u \to v} \int_1^{3 u \to v} \left[ \frac{9}{2} \ln v - \frac{v^4}{18} \right] dvdu = \left. \left[ \frac{9}{2} \ln v - \frac{v^4}{72} \right] \right|_{v=\sqrt{3}}^{v=3} - \left. \left( \frac{9}{4} \ln 3 - \frac{9}{72} \right) \right| = \frac{9}{4} \ln 3 - 1$$

or

$$\int_R xydA = \frac{1}{2} \int_1^{\sqrt{3}} \int_1^{u \to v} du \left. \int_1^{\sqrt{3} u \to v} \ln u dv \right|_{u=1}^{u=3} = \frac{1}{2} \left( \frac{9}{2} \ln 3 - 8 \right) = \frac{9}{4} \ln 3 - 1. \quad \square$$

**Problem 5.** Let $\vec{F} (x, y) = (ye^x + \sin y) \vec{i} + (e^x + x \cos y) \vec{j}$.

1. (10%) Show that $\vec{F}$ is a conservative vector field; that is, find a scalar potential $\varphi$ such that $\vec{F} = \nabla \varphi$.

2. (10%) Let $C$ be a curve given by $\vec{r}(t) = (\cos t, \sin t)$ with $0 \leq t \leq \pi$. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ by the fundamental theorem of line integrals.

**Sol:**

1. Suppose that $\vec{F} = \nabla \varphi$. Then $\phi_x = ye^x + \sin y$ and $\phi_y = e^x + x \cos y$. Therefore, $\phi(x, y) = ye^x + x \sin y + C_1(y)$ and $\phi(x, y) = ye^x + x \sin y + C_2(x)$. This implies that $C_1(y) = C_2(x)$; thus $C_1 = C_2 = \text{const}$. Therefore, if

$$\phi(x, y) = ye^x + x \sin y,$$

then $\vec{F} = \nabla \phi$ which implies that $\vec{F}$ is conservative.
2. Since $\vec{F} = \nabla \phi$ is conservative, by the fundamental theorem of line integrals,
\[ \int_C \vec{F} \cdot d\vec{r} = \phi(\vec{r}(\pi)) - \phi(\vec{r}(0)) = \phi(-1, 0) - \phi(1, 0) = 0. \]

**Problem 6.** Let $T = \{(u, v) \mid 0 \leq v \leq \frac{\pi}{2}, 0 \leq u \leq \pi - v\}$ be a triangular region on $uv$-plane, and $\vec{r}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$, $(u, v) \in T$, be a parametrization of a part of a surface $S$ on the sphere.

1. (10%) Compute $|\vec{r}_u \times \vec{r}_v|$. 

2. (10%) Compute the surface integral $\int_S y \, dS$.

**Sol:**

First we compute $\vec{r}_u$ and $\vec{r}_v$ as follows:
\[ \vec{r}_u(u, v) = (-\sin u \sin v, \cos u \sin v, 0), \]
\[ \vec{r}_v(u, v) = (\cos u \cos v, \sin u \cos v, -\sin v). \]

1. Therefore,
\[ \vec{r}_u(u, v) \times \vec{r}_v(u, v) = (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v) \]
\[ = (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v); \]

thus
\[ |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| = \sqrt{\cos^2 u \sin^4 v + \sin^2 u \sin^4 v + \sin^2 v \cos^2 v} \]
\[ = \sqrt{\sin^4 v + \sin^2 v \cos^2 v} = \sqrt{\sin^2 v} = |\sin v| = \sin v, \]

where the last equality is based on $0 \leq v \leq \frac{\pi}{2}$ which makes $\sin v$ non-negative.

2. By definition,
\[ \int_S y \, dS = \int_0^{\frac{\pi}{2}} \int_0^{\pi - v} \sin u \sin v \cdot \sin v \, du \, dv = \int_0^{\frac{\pi}{2}} \int_0^{\pi - v} \sin u \sin^2 v \, du \, dv \]
\[ = -\int_0^{\frac{\pi}{2}} \left[ \cos u \bigg|_{u=v}^{u=\pi-v} \right] \sin^2 v \, dv = 2 \int_0^{\frac{\pi}{2}} \cos v \sin^2 v \, dv = \frac{2}{3} \sin^3 v \bigg|_{v=0}^{v=\frac{\pi}{2}} = \frac{2}{3}. \]