1. Find the solution $y(t)$:
   (a) $y' - (\ln x) y = x^2$
   (b) $\frac{3y'}{x+1} + y = 2x^2$, $y(1) = 2$
   (c) $(\sin 3x) y' - (\cos 3x) y = 5$, $y(\frac{\pi}{2}) = 0$
   (d) $y'' + 2y' - 8y = 0$, $y(0) = 2$
   (e) $\frac{d^2 y}{dx^2} + 6y + 4 = 0$, $y(0) = 3$

2. Find the $n$th Taylor polynomial of $f$ at $a$.
   (a) $f(x) = \sin x$, $a = 0$
   (b) $f(x) = e^{2x}$, $a = 0$
   (c) $f(x) = \frac{1}{1+x}$, $a = 1$ 
   (d) $f(x) = \ln x$, $a = 1$

3. Find the following limit:
   (a) $\lim_{n \to \infty} (1 + \frac{5}{n^2})^n$
   (b) $\lim_{n \to \infty} \frac{n!}{3^n}$
   (c) $\lim_{n \to \infty} \frac{n^2}{n^n}$
   (d) $\lim_{n \to \infty} \frac{\ln n}{n^{0.01}}$

4. Find the arc length of $f(x)$ over $[a, b]$:
   (i) $f(x) = 2x^2$, $[a, b] = [0, 1]$
   (ii) the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$
   (iii) $f(x) = (\frac{x}{2})^4 + \frac{1}{2x^2}$, $[a, b] = [2, 4]$
   (iv) $f(x) = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}$, $[a, b] = [2, 8]$
5. Whether the infinite series is convergent or divergent? Why?

(i) \( \frac{\infty}{n=1} \frac{1}{n} \)

(iii) \( \frac{\infty}{n=1} n \cdot e^{-n^2} \)

(v) \( \frac{\infty}{n=1} \sin \frac{1}{n} \)

(Vii) \( \frac{\infty}{n=1} (-1)^n \frac{n^4}{n^3+1} \)

(ix) \( \frac{\infty}{n=3} \frac{n^3}{\sqrt{n^4-2n^2+1}} \)

(ii) \( \frac{\infty}{n=2} e^{3-2n} \)

(iv) \( \frac{\infty}{n=1} \left(1 - \cos \frac{1}{n}\right) \)

(Vi) \( \frac{\infty}{n=1} \frac{\sin n}{n^2} \)

(Viii) \( \frac{\infty}{n=2} \frac{1}{n^{15} \cdot \sin n} \)

(X) \( \frac{\infty}{n=1} \frac{(-1)^n + 1}{(2n+1)!} \)

6. A tank, in the following respectively, filled with water. Let \( y(t) \) be the water level at time \( t \).

(a) Find the differential equation satisfied by \( y(t) \) and solve the \( y(t) \).

(b) How long does it take for the tank to empty?
1. 

(a) the integrating factor: \( v(x) = e^{\int \frac{1}{x} \, dx} = e^x \)

原式 \( \Rightarrow (x^2 e^x y)’ = e^x \)

\( \therefore x^2 e^x y = e^x + C \)

Hence, \( y_1(x) = x^2 + Cx^2 e^{-x} \)

(b) the integrating factor: \( v(x) = e^{\int \frac{1}{x+1} \, dx} = x+1 \)

原式 \( \Rightarrow (x+1) y’ = x^2 + x^2 \)

\( \therefore (x+1) y = \ln x - \frac{1}{x} + C \)

Hence, \( y_2(x) = \frac{1}{x+1} (\ln x - \frac{1}{x} + C) = \frac{1}{x+1} (\ln x - \frac{1}{x} + C) \)

\( \therefore y(1) = 2 \)

\( \Rightarrow 2 - 2 = \ln 1 - 1 + C \)

\( \Rightarrow C = 5 \)

(c) 原式 \( y' - \cot(3x) \cdot y = 5 \cdot \csc 3x \)

the integrating factor: \( v(x) = e^{\int -\cot 3x \, dx} = \frac{\sin \frac{1}{3} x}{3x} \)

\( (\sin \frac{1}{3} x y)’ = 5 (\sin \frac{1}{3} x)^{\frac{1}{2}} \)
(d) Let $y = e^{ax}$

$$\Rightarrow a^2 e^{ax} + 2a e^{ax} - 8 e^{ax} = e^{ax} (a^2 + 2a - 8) = 0$$

$$\Rightarrow a^2 + 2a - 8 = 0$$

$$\Rightarrow a = -4, 2$$

$$\therefore y(x) = C_1 e^{-4x} + C_2 e^{2x}$$

$$\Rightarrow y'(0) = C_1 + 2C_2 = 2$$

$$y''(0) = -4C_1 + 2C_2 = 2$$

$$\Rightarrow C_1 = \frac{1}{2}, C_2 = \frac{5}{2}$$

Hence,

$$y(x) = \frac{1}{2} e^{-4x} + \frac{5}{2} e^{2x}$$

(e) Method

$$\Rightarrow \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = -dx$$

$$\Rightarrow \ln \left| 3y^2 + 2 \right| = -x + C$$

$$\Rightarrow \ln \left| 3y^2 + 2 \right| = -x + \ln 2$$

Hence,

$$y(x) = \pm \frac{1}{2} e^{-x} - \frac{2}{3}$$

(a) $f(x) = \sin x \Rightarrow f'(0) = 0$

$f(x) = \cos x \Rightarrow f'(1) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$$\Rightarrow T_0(x) = 0, \quad T_3(x) = T_4(x) = X - \frac{1}{3!} X^3$$

$$\Rightarrow T_1(x) = T_5(x) = X, \quad T_6(x) = X - \frac{1}{3!} X^3 + \frac{1}{5!} X^5$$

Hence, in general,

$$T_{2n+1}(x) = T_{2n+2}(x) = X - \frac{1}{3!} X^3 + \cdots + (-1)^n \frac{1}{(2n+1)!} X^{2n+1}$$
(b) \( f(x) = e^{2x} \Rightarrow f(0) = 1 \)
\[
f'(x) = 2e^{2x} \Rightarrow f'(0) = 2
\]
\[
f''(x) = 4e^{2x} \Rightarrow f''(0) = 4
\]
\[\vdots\]
\[
\Rightarrow T_0(x) = 1 \quad , \quad T_1(x) = 1 + 2x \quad , \quad T_2(x) = 1 + 2x + \frac{4}{2!}x^2
\]
\[
T_3(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 \quad , \quad \ldots
\]
Hence, in general,
\[
T_n(x) = 1 + 2x + \frac{2^2}{2!}x^2 + \ldots + \frac{2^n}{n!}x^n
\]

(c) \( f(x) = \frac{1}{1 + x} \Rightarrow f(1) = \frac{1}{2} \)
\[
f'(x) = -\frac{1}{(1 + x)^2} \Rightarrow f'(1) = -\frac{1}{2^2}
\]
\[
f''(x) = \frac{2}{(1 + x)^3} \Rightarrow f''(1) = \frac{2}{2^3}
\]
\[
f'''(x) = -\frac{6}{(1 + x)^4} \Rightarrow f'''(1) = -\frac{6}{2^4}
\]
\[\vdots\]
\[
\Rightarrow T_0(x) = \frac{1}{2} \quad , \quad T_1(x) = \frac{1}{2} - \frac{1}{4} (x-1) \quad , \quad T_2(x) = \frac{1}{2} - \frac{1}{4} (x-1) + \frac{1}{8} (x-1)^2
\]
\[
T_3(x) = \frac{1}{2} - \frac{1}{4} (x-1) + \frac{1}{8} (x-1)^2 - \frac{1}{16} (x-1)^3 \quad , \quad \ldots
\]
Hence, in general,
\[
T_n(x) = \frac{1}{2} - \frac{1}{4} (x-1) + \frac{1}{8} (x-1)^2 + \ldots + (-1)^n \frac{1}{2^n n!} (x-1)^n
\]

(d) \( f(x) = \ln x \Rightarrow f(1) = 0 \)
\[
f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1 \]
\[
f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1
\]
\[\vdots\]
\[
\Rightarrow T_0(x) = 0 \quad , \quad T_1(x) = (x-1) \quad , \quad T_2(x) = (x-1) - \frac{1}{2} (x-1)^2 \quad , \quad \ldots
\]
Hence, in general,
\[
T_n(x) = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 + \ldots + (-1)^{n+1} \frac{(x-1)^n}{n}
\]
3.

(a) \( \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^n = e \lim_{n \to \infty} \ln \left(1 + \frac{r}{n}\right)^n \)

\[ = e \lim_{n \to \infty} \ln \left(1 + \frac{r}{n}\right) \]

\[ = e \]

(b) \( \frac{n}{3^n} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{3 \cdot 3 \cdot 3 \cdot \ldots \cdot 3} = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{3} \cdot \left(\frac{4}{3} \cdot \frac{5}{3} \cdot \ldots \cdot \frac{n-1}{3}\right) \cdot \frac{2n}{3^2} \)

\[ \Rightarrow \frac{n}{3^n} > \frac{2n}{3^2} \]

\[ \Rightarrow \lim_{n \to \infty} \frac{2n}{3^2} = \infty \]

Hence, \( \lim_{n \to \infty} \frac{n}{3^n} = \infty \)

(c) \( \lim_{n \to \infty} n^{\frac{r}{n}} = e \lim_{n \to \infty} \frac{\ln n}{n} \)

\[ = e \]

(d) \( \lim_{n \to \infty} \frac{\ln n}{n^{\alpha-1}} = \lim_{n \to \infty} \frac{1}{\alpha-1} n^{1-\alpha} = 0 \)
4.

(i) \( \text{Arc length of } f(x) = 2x^2 \text{ on } [0, 1] \)
\[
= \int_0^1 \sqrt{1 + (4x)^2} \, dx
\]
\[
= \left[ \frac{1}{4} \ln \left( \sqrt{1 + (4x)^2} + 4x \right) \right]_0^1
\]
\[
= \frac{1}{4} \ln \left( \sqrt{17} + 4 \right) - \frac{1}{4} \ln \left( 1 + 4 \cdot 0 \right)
\]
\[
= \frac{1}{4} \ln \left( \sqrt{17} + 4 \right)
\]

Hence, arc length = \( \frac{1}{4} \ln (\sqrt{17} + 4) \)

(ii) \( \text{Arc length of } f(x) = (\frac{x}{2})^4 + \frac{1}{2x^2} \text{ on } [2, 4] \)
\[
= \int_2^4 \sqrt{\left( \frac{x}{2} \right)^4 \left( 1 \right) + \left( \frac{1}{2x} \right)^2 \left( -4 \right)} \, dx
\]
\[
= \int_2^4 \frac{x^4 + 1}{2x^2} \, dx
\]
\[
= \left[ \frac{1}{16} x^4 - \frac{1}{2} \right]_2^4
\]
\[
= \frac{4 \cdot 4^4 - 2}{32} - \left( \frac{1}{16} \cdot 2^4 - \frac{1}{2} \right)
\]
\[
= \frac{64 - 2}{32} - \frac{1}{2} + \frac{1}{2}
\]
\[
= \frac{31}{16}
\]
(iv) Arc length of \( f(x) = \frac{1}{2}x^{1/2} - x^{1/4} \) on \([3, 8]\)

\[
\int_3^8 \sqrt{\left(\frac{1}{2}x^{1/2} + \frac{1}{4}x^{-1/4}\right)^2} \, dx
\]

\[
\int_3^8 \frac{1}{2}x^{1/2} + \frac{1}{4}x^{-1/4} \, dx
\]

\[
\frac{17\sqrt{2}}{3}
\]

5.

(i) Reference III.3 Example 1.

(ii) \[
\frac{\alpha}{\ln x} e^{x-n} = \frac{\alpha}{\ln x} e^{x} \cdot \left(\frac{1}{e^x}\right)^n
\]

\[\therefore \frac{1}{e^x} < 1\]

\[\therefore \frac{\alpha}{\ln x} e^{x-n} = \frac{e}{e^x-1} \text{ converges.}\]

(iii) Let \( f(x) = xe^{-x^2}, \quad x > 1 \)

\[\Rightarrow f'(x) = (1 - 2x^2)e^{-x^2} < 0, \quad \text{for } x > 1 \]

\[\Rightarrow f(x) \text{ is decreasing for } x > 1 \]

\[\int_3^\infty xe^{-x^2} \, dx = \lim_{a \to \infty} \int_3^a xe^{-x^2} \, dx = -\frac{1}{2} \lim_{a \to \infty} (e^{-x^2} - e^{-9}) = \frac{1}{2} e^{-4}\]

\[\therefore \frac{\pi}{\ln 2} e^{-n^2} \text{ converges.}\]

(iv) \[
\lim_{n \to \infty} \frac{1 - \cos \frac{1}{n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{-\frac{1}{n^2} \sin \frac{1}{n}}{-\frac{2}{n^3}} = \frac{1}{2} > 0, \quad \therefore \lim_{n \to \infty} \frac{1 - \cos \frac{1}{n}}{\frac{1}{n^2}} = 1.
\]

\[\frac{\frac{18}{\pi}}{\frac{1}{n^2}} \text{ converges.}\]

By the limit comparison test we know

\[\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right) \text{ converges.}\]
(v) \( \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 > 0 \) converges.

by the limit comparison test we know

\( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

(vi) \( \frac{\sin n}{n^2} \leq \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

\( \frac{\sin n}{n} \) converges absolutely.

\( \Rightarrow \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges.

* Note: conv. \( \Rightarrow \) conv.

(vii) \( \lim_{n \to \infty} \frac{n^4}{n^2 + 1} = \infty. \)

by the divergence test we know

\( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \) diverges.

(viii) let \( f(x) = \frac{1}{x^{1/2} \cdot \ln x}, \quad x > 2 \)

\( \Rightarrow f'(x) = -\frac{1}{2x^{3/2} \cdot (x \cdot (\ln x)^{-1})} < 0, \quad \text{for} \quad x > 2 \)

\( \Rightarrow f(x) \) is decreasing for \( x > 2 \).

\( S_2^\infty \frac{1}{x^{1/2} \cdot \ln x} \, dx = \lim_{a \to \infty} S_2^a \frac{1}{x^{1/2} \cdot \ln x} \, dx \)

\( \Rightarrow \frac{1}{x^{1/2}} > \frac{1}{x}, \quad \text{for} \quad x > 1 \)

\( \Rightarrow \frac{1}{x^{1/2} \cdot \ln x} > \frac{1}{x \cdot \ln x}, \quad \text{for} \quad x > 1 \)

\( S_2^\infty \frac{1}{x^{1/2} \cdot \ln x} \, dx = S_{x=1}^{\infty} \frac{1}{u} \, du, \quad u = \ln x \)

\( = \ln (\ln x)|_{x=2}^{\infty} \)

\( = \infty \)

\( \Rightarrow S_2^\infty \frac{1}{x^{1/2} \cdot \ln x} \, dx \) diverges.

by the integral test we know

\( \sum_{n=2}^{\infty} \frac{1}{n^{1/2} \cdot \ln n} \) diverges.
(ix) \( \frac{n^3}{\sqrt{n^2 - 2n + 1}} = \frac{n^3}{n^2 - 1} \), for \( n \geq 3 \)

\[ \lim_{n \to \infty} \frac{n^3}{n^2 - 1} = 1 > 0 \]

\( \sum_{n=3}^{\infty} n \) diverges.

by the limit comparison test, we know

\( \sum_{n=3}^{\infty} \frac{n^3}{n^2 - 1} \) diverges.

(x) let \( a_n = \frac{1}{(2n+1)!} \)

\( \{a_n\} \) is a decreasing sequence, and \( \lim_{n \to \infty} a_n = 0 \)

by the Leibniz test we know

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \] converges.

6.

\( x \left( \frac{dy}{dx} \right) = -\frac{8B \sqrt{y}}{A y^3} \)

(a) \( \frac{r}{\sqrt{2}} = \frac{4}{\sqrt{2}} \) \( \Rightarrow r = \frac{4}{\sqrt{2}} \) \( \Rightarrow A(y) = \frac{4}{\sqrt{2}} y^4 \)

\[ B = 4 \pi \sin = \frac{\pi}{3} \] \( \Rightarrow ft^2 \)

\[ \therefore \frac{dy}{dt} = -2 y^{\frac{1}{2}} \]

\[ 2 \left( \frac{x}{5} \right)^{\pi} + \left( y - x \right)^{\pi} = 400 \] \( \Rightarrow x + 2 \sqrt{400 - y^2} \) \( \Rightarrow A(y) = 120 \sqrt{400 - y^2} \)

\[ B = \frac{16 \pi \sin}{\frac{\pi}{9}} \] \( \Rightarrow ft^2 \)

\[ \therefore \frac{dy}{dt} = -\frac{\pi}{135 \sqrt{400 - y^2}} \]

(b) \( \frac{dy}{dt} = -2 \sqrt{y} \) \( \Rightarrow \frac{dy}{\sqrt{y}} = -2 t + C \)

\( \therefore y(0) = 12 \) \( \Rightarrow C = \frac{25}{3} \) \((12)^{\frac{5}{2}}\)

\( \Rightarrow y(t) = \left( \frac{25}{3} \right)^{\frac{5}{2}} - 5 t \)

\( \frac{dy}{dt} = 0 \) \( \Rightarrow t = \frac{1}{5} \) \((12)^{\frac{5}{2}}\)

\[ \frac{dy}{dt} = -\frac{\pi}{135 \sqrt{400 - y^2}} \] \( \Rightarrow -\frac{2}{3} \left( 40 - y^2 \right)^{\frac{5}{2}} = -\frac{\pi}{135} y + C \)

\( \therefore y(0) = 40 \) \( \Rightarrow C = 0 \)

\( \Rightarrow y(t) = 40 - \left( \frac{25}{9} t \right)^{\frac{5}{2}} \)

\( y(t) = 0 \) \( \Rightarrow t = \frac{90}{\pi} \) \((40)^{\frac{1}{2}}\)