(1) (15%)

(i) State the respectively following definition of the Lebesgue integral on \( \mathbb{R}^d \): Simple functions; Bounded functions supported on a set of finite measure; Non-negative function; Integrable functions (the general case).

(ii) Prove that: The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.

(iii) (Chebychev inequality) Suppose \( f \geq 0 \), and \( f \) is integrable. If \( \alpha > 0 \) and \( E_\alpha = \{ x : f(x) > \alpha \} \), prove that \( m(E_\alpha) \leq \frac{1}{\alpha} \int f \).

(2) (20%) State and prove the following theorem respectively: Bounded convergence theorem; Fatou’s lemma; Monotone convergence theorem; Lebesgue dominated convergence theorem.

(3) (20%) Prove that:

(i) The space \( \mathbb{R}^d \), \( d \geq 1 \), is a complete vector space in its usual norm.

(ii) The space \( L^1 \) is a complete vector space in its metric.

(4) (10%) Prove that if \( f \) is finite almost everywhere, then \( f \) is integrable if and only if
\[
\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \quad \text{if and only if} \quad \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.
\]

(ii) (10%) Let
\[
f(x) = \begin{cases} |x|^{-a} & \text{if } |x| > 1, \\ 0 & \text{otherwise} \end{cases}
\]

Then prove that \( f \) is integrable on \( \mathbb{R}^d \) if and only if \( a > d \).

(5) (5%) Prove that if \( f \) is uniformly continuous on \( \mathbb{R} \) and integrable, then \( \lim_{x \to \infty} f(x) = 0 \).

(ii) (5%) Prove that if \( f \) is integrable on \( \mathbb{R}^d \), real-valued and \( \int_E f(x) \, dx \geq 0 \) for every measurable \( E \), then \( f(x) \geq 0 \) a.e \( x \).

(iii) (5%) If \( f \) is integrable on \( \mathbb{R} \), show that
\[
F(x) = \int_{-\infty}^{x} f(t) \, dt
\]
is uniformly continuous.

(6) (10%) Prove or disprove that if \( f \in L^1(\mathbb{R}^d) \) and a sequence \( \{ f_n \} \subset L^1(\mathbb{R}^d) \) such that \( \| f_n - f \|_{L^1} \to 0 \), then \( f_n(x) \to f(x) \) a.e. \( x \).