Some Qualitative Properties of the Riemann Problem in Gas Dynamical Combustion

Cheng-Hsiung Hsu and Song-Sun Lin*

Department of Applied Mathematics, National Chiao-Tung University, Hsin-Chu 30050, Taiwan

Received August 19, 1996

We study the Chapman–Jouguet (CJ) model and the selfsimilar Zeldovich–von Neumann–Döring (SZND) model in chemically reacting gas flows. We discover some basic relationships among ignition temperature $T_i$, total chemical binding energy $Q$, and the adiabatic exponent $\gamma$ of polytropic gas. From these relations, we can determine when temperatures along the SZND burning solutions are higher than the ignition temperature $T_i$. We also study the all possible selfsimilar solutions for the SZND-model. From these results, we can determine when selfsimilar solutions for the CJ-model are the limits of selfsimilar solutions of the SZND model when the reaction rate tends to infinity.

1. INTRODUCTION

Two well-known mathematical models have been used frequently to study combustion phenomena in chemically reactive gas flows: the Chapman–Jouguet (CJ) model and the Zeldovich–von Neumann–Döring (ZND) model (see [2, 10]). In Lagrangian coordinates, the CJ-model is expressed as

$$
\begin{align*}
&u_t + p_x = 0, \\
&\tau_t - u_x = 0, \\
&E_t + (pu)_x = 0, \\
&q(x, t) = \begin{cases} 0 & \text{if } \sup_{0 \leq s \leq t} T(x, s) > T_i; \\
q(x, 0), & \text{otherwise,}
\end{cases}
\end{align*}
$$

where $u, p, \tau, T, T_i, q$, and $E$ are respectively velocity, pressure, specific volume, temperature, ignition temperature, chemical binding energy, and specific total energy. More precisely, $E = (1/2) u^2 + e + q$, where $e$ is the

* The work was supported in part by the National Science Council of the Republic of China with Project NSC84-2121-M009-004.
internal energy. For polytropic gases, the internal energy $e = e(T) = pT/(\gamma - 1)$, where $\gamma$ is the adiabatic exponent with $\gamma \in (1, 5/3)$ for media occurring. Temperature $T$ satisfies Boyle and Gay-Lussac's law, $pT = RT$. $R$ is a constant that depends on the effective weight of particular gases. For simplicity, we assume $R$ and $\gamma$ remain unchanged during the reaction, and that $R = 1$. The CJ-model is based on two physical assumptions:

(i) the reaction rate is infinitely large (i.e., the reaction zones are infinitely thin),

(ii) the effects of viscosity and heat conduction are negligible.

In the ZND-model, on the other hand, a finity but large reaction rate is assumed although the effects of viscosity and heat conduction are still ignored. The ZND-model is expressed as

\[
\begin{aligned}
\begin{cases}
\partial_t u + p_x = 0, \\
\tau - u_t = 0, \\
E_t + (pu)_x = 0, \\
q_t = -k\phi(T)q,
\end{cases}
\end{aligned}
\]

where

\[
\phi(T) = \begin{cases}
1 & \text{if } T > T_i, \\
0 & \text{if } T \leq T_i,
\end{cases}
\]

and $k$ is a positive constant related to the reaction rate. It is natural to ask whether or not the CJ-model is a limit of the ZND-model as $k \to \infty$. The question is still unsolved due to the mathematical difficulty of obtaining the existence of global (in time) weak solutions for the ZND-model and then studying their asymptotic behavior as $k \to \infty$.

However, Ying and Terng [11] studied the following simplified scalar combustion model

\[
\begin{aligned}
\begin{cases}
(u + Qz)_t + f(u)_x = 0, \\
zt = -k\phi(u)z.
\end{cases}
\end{aligned}
\]

They were able to prove the existence and uniqueness of a solution for the Riemann problem. Furthermore, they proved the existence of limits on the solutions as $k \to \infty$ and found that the limit function is a solution of the Riemann problem for the corresponding scalar CJ-model. Later, Jäger, Yang and Zhang [4] studied the following selfsimilar scalar ZND-model

\[
\begin{aligned}
\begin{cases}
(u + qz)_t + f(u)_x = 0, \\
z_t = -\frac{k}{l} \phi(u)z.
\end{cases}
\end{aligned}
\]
They proved that all selfsimilar solutions for the scalar CJ-model are the limits of the solutions of the Riemann problem stated in (A2). Based on these results, Tan and Zhang [8] then studied the following selfsimilar ZND (SZND) model

\[
\begin{align*}
  u_t + p_x &= 0, \\
  \tau_t - u_x &= 0, \\
  E_t + (pu)_x &= 0, \\
  q_t &= -\frac{k}{\gamma} \varphi(t) q.
\end{align*}
\]

\((\text{SZND})\)

Previously, Courant and Friedrichs [2] proved that any combustional shock wave (deflagration and detonation) must satisfy Jouguet’s rule. However, only quite recently has a complete solution that satisfies Jouguet’s rule for the Riemann problem as it relates to the CJ-model been obtained by Zhang and Zheng [13]. The number of solutions may be (at most) nine for some initial data. Later, Tan and Zhang [8] proved that these selfsimilar solutions for the CJ-model are limits of the SZND solutions as \(k \to \infty\) assuming the following:

- (TZ-1) the selfsimilar solutions for the SZND-model are of a special type; (we call them simple types in this paper),
- (TZ-2) temperatures along the SZND burning solutions are higher than the ignition temperature \(T_i\).

Due to the discontinuity of \(\varphi\) at \(T_i\), initial-value problems involving selfsimilar solutions for the SZND model may yield non-unique results at \(T = T_i\). In this paper, we discuss this issue in detail and identify the solutions obtained in [8] as simple solutions.

We discovered the answers for assumption (TZ-2), lie in the relationships among ignition temperature \(T_i\), total chemical binding energy \(Q\), and the adiabatic exponent \(\gamma\) of polytropic gas. More precisely, it depends on the relation between \(T_i\) and \(Q_\ell(\gamma) = ((1 - 9\mu^4)/2\mu^2)Q\) with \(\mu^2 = (\gamma - 1)/(\gamma + 1)\). These are intrinsic properties of the CJ-model. We then divide all unburnt states into three classes: A, B and C. (TZ-2) is always true for the Jouguet diagrams of class A unburnt states (see Section 2 for further details). (TZ-2) is only partially true for the Jouguet diagrams associated with class B and C unburnt states. From these observations, we can determine when selfsimilar solutions for the CJ-model are the limits of simple solutions of the SZND-model.

Since many studies of combustion theory exist, we mention only those few that are closely related to our work.
Based on the work of Ying and Terng [11], Liu and Zhang [6] obtained a set of entropy conditions for the scalar CJ-model. This sect of entropy conditions consists of two parts—pointwise and global, and they were able to prove the existence and uniqueness of solutions. In [7], Majda studied the combustion profile of scalar combustion model with finite reaction rate and diffusion. Using singular perturbation methods, Wagner [9] and Gasser and Szmolyan [3] studied combustional problems involving low viscosity, heat induction and diffusion. In [1], Chen proved the existence of global generalized solutions to the compressible Navier-Stokes equation for a reacting mixture with discontinuous Arrhenius functions.

The paper is organized as follows. In Section 2, we study (CJ) and obtain some new properties of it, including some relationships between $T_i$, $Q$ and $\#$. We then divide all unburnt states into three classes in order to study temperatures along the burning solution. In Section 3, we study (SZND) and establish the existence of global selfsimilar solutions. The solutions at $T = T_i$ are classified as simple and non-simple. The simple solutions are used to approximate (CJ) as $k \to \infty$. In Section 4, after improving the strength of the results obtained by Tan and Zhang [8], we derive a complete answer when the CJ-model is a limit of the SZND-model as $k \to \infty$.

2. SOME BASIC PROPERTIES OF THE CJ-MODEL

In this section we shall review some known properties of the CJ-model and provide some new results which are interesting in themselves and also useful in studying the SZND-model.

For a given burnt state $(u_0, p_0, \tau_0, 0)$, all states $(u, p, \tau, 0)$ that can be linked by shock (S) or rarefaction waves (R) to $(u_0, p_0, \tau_0, 0)$ are given by

$$
R: \quad p\tau^2 = p_0\tau_0^2, \quad 0 < p \leq p_0, \quad (2.1)
$$
$$
u = u_0 - \frac{1 - \mu^4}{\mu^4} \tau_0^2 \frac{1}{p_0^{1/2}} \left( p^{(\gamma - 1)/2} - p_0^{(\gamma - 1)/2} \right), \quad (2.2)
$$

$$
S: \quad (p + \mu^2 p_0) \left( \tau - \mu^2 \tau_0 \right) = (1 - \mu^4) p_0 \tau_0, \quad p > p_0, \quad (2.3)
$$
$$
u = u_0 - (p - p_0) \left( 1 - \mu^2 \right) \left( 1 + \mu^2 \right) \left( p + \mu^2 p_0 \right)^{(1/2)} \tau_0, \quad (2.4)
$$

where $\mu^2 = (\gamma - 1)/(\gamma + 1)$.

Here, Lax entropy conditions are assumed, see [5]. However, for a given unburnt state $(u_0, p_0, \tau_0, Q)$, in addition to the non-combustional shock...
waves (S) and rarefaction waves (R) \((u, p, \tau, Q)\) given \((2.1) \sim (2.4)\), we also have combustional shock waves: detonation waves (DT) and deflagration waves (DF) \((u, p, \tau, 0)\) that can be linked to \((u_0, p_0, \tau_0, Q)\) and lie on a Hugoniot curve:

\[
(p + \mu^2 p_0)(\tau - \mu^2 \tau_0) = (1 - \mu^4) p_0 \tau_0 + 2 \mu^2 Q. \tag{2.5}
\]

\(DT \equiv DT((\tau_0, p_0))\) is the upper portion of the curve, i.e., \(p \geq p_A\), where

\[
p_A = p_0 + \frac{\chi^2}{(1 - \mu^2) \tau_0} \tag{2.6}
\]

and \(\chi = (2 \mu^2 Q)^{1/2}\). DF \(\equiv DF((\tau_0, p_0))\) is the lower portion of the curve, i.e., \(p \leq p_0\). Furthermore, there is a unique Rayleigh line

\[
- \eta_*^2 = \frac{p - p_0}{\tau - \tau_0}, \quad \eta_* > 0, \tag{2.7}
\]

which starts from \((\tau_0, p_0)\) and is tangent to DT at point \((\tau_*, p_*)\). \((\tau_*, p_*)\) is called the Chapman–Jouguet detonation point (CJDT) of state \((\tau_0, p_0)\).

Moreover, the CJDT point \((\tau_*, p_*)\) divides DT into two parts: strong detonation (SDT) for \(p > p_*\) and weak detonation (WDT) for \(p_A \leq p < p_*\). Similarly, there is a Chapman–Jouguet deflagration point (CJDF) \((\tau^*, p^*)\) on DF that divides DF into two parts: weak deflagration (WDF) for \(0 < p < p^*\) and strong deflagration (SDF) for \(0 < p < p^*\). For details, see [2, 8, 13] and Fig. 1.

For any \(\eta \in [\eta_*, \infty)\), the associated Rayleigh line

\[
- \eta^2 = \frac{p - p_0}{\tau - \tau_0} \tag{2.8}
\]

intersects SDT at \((\tau(\eta), p(\eta))\) and \(S\) at \((\tau^*(\eta), p^*(\eta))\) uniquely. \((\tau^*(\eta), p^*(\eta))\) is called the von Neumann point. They are very important in our study of the CJ-model and the explicit expressions of these points are given below.

The Riemann problem stated below has material in a burnt state on the left and material in an unburnt state on the right,

\[
\text{burnt state} (-): (u_-, p_-, \tau_-, 0) \quad \text{for} \quad x < 0,
\]

and

\[
\text{unburnt state} (+): (u_+, p_+, \tau_+, Q) \quad \text{for} \quad x > 0,
\]
The state $(\sigma)$ of Hugoniot curve $S(\sigma) \cup R(\sigma)$ is given by (2.1)~(2.4) with $u_0, p_0$ and $\sigma$ being replaced by $u_-, p_-$, and $\tau_-$, respectively. Courant and Friedrichs [2] pointed out that the WDT and SDF are not stable for the unburnt state $(\sigma)$. If we assume as in [8, 13] that the temperature at the front WDF bank is exactly at the ignition temperature $T_i$, then Jouguet’s rule implies three different kinds of wave series can be linked to state $(\sigma)$:

(i) $S(+)$ or $R(+)$ (with $q = Q$) (containing no combustion waves);
(ii) $(i) + \text{WDF}(i)$ or $(i) + \text{CJDF}(i) + R(\text{CJDF}(i))$ (containing no DT waves); and
(iii) $\text{SDT}(\sigma)$ or $\text{CJDT}(\sigma) + R(\text{CJDT}(\sigma))$.

Here $i \equiv i(\sigma) \equiv (u_i, p_i, \tau_i, Q) \equiv (u(\sigma), p(\sigma), \tau(\sigma), Q)$ is the state at $S(\sigma)$ with ignition temperature $T_i$. (The symbol “$+$” between two states

![Fig. 1. Chapman–Jouguet’s diagram.](image)
in (ii) and (iii) means “followed by”. Substituting states (+) and (i) into (2.1) and (2.3), we obtain the following expressions.

\[ R(+) : \quad p\tau^i = p_+ \tau_+^i, \quad p < p_+. \] (2.9)

\[ S(+) : \quad (p + \mu^2 \rho_+)(\tau - \mu^2 \tau_+) = (1 - \mu^4) p_+ \tau_+, \quad p > p_+. \] (2.10)

\[ SDT(+) : \quad (p + \mu^2 p_+)(\tau - \mu^2 \tau_+) = (1 - \mu^4) p_+ \tau_+ + 2\mu^2 Q, \quad p > p_c. \] (2.11)

\[ WDF(i) : \quad (p + \mu^2 \rho_i)(\tau - \mu^2 \tau_+) = (1 - \mu^4) T_+ + 2\mu^2 Q, \quad p_i^c < p < p_c. \] (2.12)

\[ R(CJDT(+)) : \quad p\tau^i = p_c \tau_c^i, \quad p < p_c. \] (2.13)

\[ R(CJDF(i)) : \quad p\tau^i = p_c^i \tau_c^i, \quad p_i^c < p < p_c. \] (2.14)

Hence the set of all possible states can be linked to state (+) is

\[ J(+) \equiv R(+) \cup S(+) \cup JDT(+) \cup JDF(+), \]

where

\[ JDT(+) \equiv SDT(+) \cup R(CJDT(+)) \]

and

\[ JDF(+) \equiv WDF(i) \cup R(CJDF(i)). \]

\( J(+) \) is called the Jouguet diagram just because it relates to Jouguet’s rule [2, 8, 13]. A typical Jouguet diagram is given in Fig. 2.

From the above observations, there are at most three solutions for our Riemann problem, see [8, 13]. Our main task in this paper is to study which one of them can be a limit of solutions for the SZND-model as \( k \to \infty \).

From now on, for simplicity, we shall omit dependence on (+) wherever the omission does not cause any confusion. We first establish some explicit expressions for CJDT(+) and CJDF(+).

**Proposition 2.1.** The coordinates (\( \tau_c, p_c \)) and (\( \tau^c, p^c \)) of CJDT(+) and CJDF(+) are given by

\[ p_c = p_+ + \frac{\alpha(x + \beta)}{(1 - \mu^2) p_+}, \quad \tau_c = \tau_+ + \frac{\alpha(x - \beta)}{(1 + \mu^2) p_+}, \] (2.15)
and
\[ p^e = p_+ + \frac{x(x - \beta)}{(1 - \mu^2) \tau_+}, \quad \tau^e = \tau_+ + \frac{x(x + \beta)}{(1 + \mu^2) p_+}, \quad (2.16) \]

where
\[ \alpha = (2\mu^2 Q)^{1/2} \quad \text{and} \quad \beta = \beta(+ = \{2\mu^2 Q + (1 - \mu^4) p_+ \tau_+\}^{1/2}. \quad (2.17) \]

Furthermore,
\[ T_c \equiv T_c(+) \equiv p_+ \tau_c = p_+ \tau_+ + \frac{2x(x + \mu^2 \beta)}{1 - \mu^2} - \alpha^2. \quad (2.18) \]

Proof. On SDT(+), taking \( p = p(\tau) \), it is easy to verify that
\[ \frac{dp}{d\tau} = \frac{p + \mu^2 p_+}{\tau - \mu^2 \tau_+}. \quad (2.19) \]
Since CJDT(+) is the point of tangency between the Rayleigh line and the Hugoniot curve (2.11). \((\tau_+, p_+)\) satisfies

\[
\frac{p_c - p_+}{\tau_c - \tau_+} = \frac{p_c + \mu^2 p_+}{\tau_c - \mu^2 \tau_+}.
\]  

(2.20)

From (2.11) and (2.20), after a direct computation we can obtain

\[
\tau_+ = (1 + \mu^2) p_+ \tau_+ / 2p_+ - (1 - \mu^2) p_+
\]

and

\[
(1 - \mu^2) \tau_+^2 + (2(\mu^2 - 1) p_+ \tau_+ - 2x^2) p_+ + (1 - \mu^2) p_+ (p_+ \tau_+ + x^2) = 0.
\]

Then (2.15) follows from the last two equations.

We can obtain (2.16) in similar fashion, thus the details are omitted, and the proof is complete.

Next, it is useful to distinguish SDT(+) from WDT(+) by comparing \(\eta\) with \(\gamma p/\tau\) as follows.

**Lemma 2.2.** If \((\tau, p) \in DT(+)\) and \(\eta\) satisfies

\[
-\eta^2 = \frac{\gamma p - p_+}{\tau - \tau_+},
\]  

(2.21)

then, on SDT(+), we have \(\eta^2 < \gamma p/\tau\), at CJDT(+), we have \(\eta^2 = \gamma p/\tau\), and on WDT(+), we have \(\eta^2 > \gamma p/\tau\). Similarly, if \((\tau, p) \in DF(+)\) and \(\eta\) satisfy (2.21), then on WDF(i), we have \(\eta^2 < \gamma p/\tau\), at CJDF(i), we have \(\eta^2 = \gamma p/\tau\), and on SDF(i), we have \(\eta^2 > \gamma p/\tau\).

**Proof.** On DT(+), taking \(p = p(\tau)\) and \(\eta = \eta(\tau)\) on (2.21), we may define

\[
g(\tau) = \gamma p(\tau) - \eta^2(\tau) \tau
\]

on \((\mu^2 \tau_+, \tau_+).\) Then it is easy to verify that

\[
\frac{dg}{d\tau} = \left[(\tau - \tau_+) + (\tau - \mu^2 \tau_+)^2\right] \frac{p + \mu^2 p_+}{(1 - \mu^2)(\tau_+ - \tau)} - \frac{p - p_+}{(1 - \mu^2)(\tau - \tau_+)} = 0,
\]
which is strictly negative in \((\mu^2 \tau_+ , \tau_+ )\). Now, (2.20) and (2.21) implies

\[
\eta_+^2 = \frac{\tau_+}{p_+}
\]

i.e., \(g(\tau_+) = 0\). Hence, the result follows for \(\text{DT}(+)\). We can obtain the results for \(\text{DF}(+)\) in similar fashion, and the proof is complete.

Explicit von Neumann points can also be computed as follows.

**Lemma 2.3.** For each \(\eta \in [\eta_0(+) , \infty)\) the von Neumann point \((\tau^\eta(\eta), p^\eta(\eta))\) is given by

\[
p^\eta(\eta) = p(\eta) + \frac{\chi^2}{\tau_+ - \tau(\eta)}, \quad (2.22)
\]

and

\[
\tau^\eta(\eta) = \tau(\eta) + \frac{\chi^2}{p_+ - p(\eta)}, \quad (2.23)
\]

where \((\tau(\eta), p(\eta))\) is the associated \(\text{SDT}(+)\) point. Furthermore, the temperature at the associated von Neumann point \((\tau^\eta_c, p^\eta_c)\) with \(\text{CJDT}\) point \((\tau_+, p_+)\) is

\[
T^\eta_c(+) = p^\eta_c \tau^\eta_c = p_+ \tau_+ + \frac{4\chi^2(\lambda + \mu^2\beta)}{1 - \mu^2} - 4\chi^2. \quad (2.24)
\]

**Proof.** Since \((\tau^\eta(\eta), p^\eta(\eta)) \in S (+)\) and

\[
-\eta^2 = \frac{p^\eta - p_+}{\tau^\eta - \tau_+}, \quad (2.25)
\]

we can easily obtain

\[
p^\eta = p_+ - \eta^2(\tau^\eta - \tau_+)
\]

and

\[
\eta^2(\tau^\eta)^2 - (1 + \mu^2)(\eta^2 \tau_+ + p_+) \tau^\eta + (1 + \mu^2) p_+ \tau_+ + \mu^2 \eta^2 \tau_+^2 = 0.
\]

From last two equations, we have

\[
p^\eta(\eta) = (1 - \mu^2) \eta^2 \tau_+ - \mu^2 p_+, \quad (2.26)
\]
and

\[ \tau^\varepsilon(\eta) = (1 + \mu^2) \frac{P_\pm}{\eta^2} + \mu^2 \tau_+ \]  \hspace{1cm} (2.27)

Substituting (2.21) into (2.26) and (2.27), (2.22) and (2.23) then follow.

Finally, let \( \eta = \eta_\varepsilon(+) \). We then have

\[ T^\varepsilon(+) = \left( p_\varepsilon + \frac{\alpha^2}{\tau_+ - \tau_\varepsilon} \right) \left( \tau_\varepsilon + \frac{\alpha^2}{p_\varepsilon - p_\varepsilon} \right). \]  \hspace{1cm} (2.28)

(2.24) follows from (2.15) and (2.28). The proof is complete.

For convenience, we denote the set of all von Neumann points of state (+) as

\[ \text{VN}(+) \equiv \{ (\tau^\varepsilon(\eta), p^\varepsilon(\eta)) \mid \eta \in [\eta_\varepsilon(+), \infty) \}. \]

From the explicit expressions above, we can easily recover the following well-known result: SDT is composed of a shock wave followed by a weak deflagration wave.

**Proposition 2.4.** For \( \eta \in [\eta_\varepsilon, \infty) \), let \( (\tau(\eta), p(\eta)) \in \text{SDT}(+) \) and the associated von Neumann point \( (\tau^\varepsilon(\eta), p^\varepsilon(\eta)) \in \text{VN}(+) \). Then \( (\tau(\eta), p(\eta)) \in \text{WDF}(\tau^\varepsilon(\eta), p^\varepsilon(\eta)) \).

**Proof.** \((\tau^\varepsilon(\eta), p^\varepsilon(\eta))\) and \((\tau(\eta), p(\eta))\) satisfy (2.10) and (2.11) respectively, i.e., we have

\[ (p^\varepsilon + \mu^2 p_+) (\tau^\varepsilon - \mu^2 \tau_+) = (1 - \mu^4) p_+ \tau_+, \]

and

\[ (p + \mu^2 p_+) (\tau - \mu^2 \tau_+) = (1 - \mu^4) p_+ \tau_+ + 2 \mu^2 Q. \]

On the other hand, both of them also lie on the same Rayleigh line with slope \(-\eta^2\), i.e.,

\[ \frac{p^\varepsilon - p_+}{\tau - \tau_\varepsilon} = \frac{p - p^\varepsilon}{\tau - \tau} = -\eta^2. \]

By direct computation, we can obtain

\[ (p + \mu^2 p^\varepsilon)(\tau - \mu^2 \tau^\varepsilon) = (1 - \mu^4) p^\varepsilon \tau^\varepsilon + 2 \mu^2 Q. \]
Since $p < p^*$ and $\tau > \tau^*$, we have $(\tau, p) \in DF((\tau^*, p^*))$. Furthermore, $\eta^2 < \gamma p/\tau$ and Lemma 2.2 imply $(\tau, p) \in WDF((\tau^*, p^*))$. The proof is complete.

Usually, the Hugoniot curve $SDF(+) \text{ is parametrized by } \tau \in [\mu^2 \tau^*, \tau^*]$. However, it can also be parametrized by $\eta \in [\eta^2(\tau), \infty]$ through (2.21). In particular, as in (2.22) and (2.23), the von Neumann set $VN(\tau)$ is already parametrized by $\eta \in [\eta^2(\tau), \infty]$. We can also use (2.21) to parametrize the von Neumann set $VN(\tau)$ by $\tau \in [\mu^2 \tau^*, \tau^*]$ whenever we like. From now on, any quantity defined on $SDT(\tau)$ or $VN(\tau)$ can be seen either as a function of $\tau$ or $\eta$. For example, the temperature $T = p\tau$ can be seen either as $T(\tau) = p(\tau)\tau$ or $T(\eta)$.

It is known that temperatures along $S(\tau)$ or $SDT(\tau)$ strictly decrease as $\tau$ increases [12].. Indeed, we have:

**LEMMA 2.5.** On $S(\tau)$ and $SDT(\tau)$, we have

\[ \frac{dT}{d\tau} < 0. \quad (2.29) \]

Furthermore, $T$ and $T^*$ strictly increase in $(\eta^2, \infty)$, with

\[ T^*(\tau^*) > T^*. \quad (2.30) \]

**Proof.** By (2.19), we have

\[ \frac{dT}{d\tau} = -\mu^2 \left( \frac{p\tau + \rho \tau}{\tau - \mu^2 \tau^*} \right). \]

Hence, (2.29) follows. From (2.29) and $d\tau/d\eta < 0$, $T$ and $T^*$ strictly increase in $(\eta^2, \infty)$. Finally, (2.30) follows by (2.28). The proof is complete.

**DEFINITION 2.6.** Given $Q > 0$ and $\gamma \in (1, 2)$, define

\[ Q^* = Q^*_\gamma = \frac{1 - 9\mu^2}{2\mu^2} Q. \quad (2.31) \]

The positive quantity $Q^*$ plays a very important role in studying the temperatures $T$ and $T^*$ along $SDT(\tau)$ and $VN(\tau)$, respectively. Indeed, we have following theorem.
Theorem 2.7. Let $Q$ and $T_+ > 0$ be given. Then,

(i) $T_+ < Q_*$ if and only if $T^*_{c}(+) < T_c(+)$. (2.32)

(ii) $T_+ = Q_*$ if and only if $T^*_{c}(+) = T_c(+)$. (2.33)

(iii) $T_+ > Q_*$ if and only if $T^*_{c}(+) > T_c(+)$. (2.34)

Furthermore, in (i) and (ii),

$T(\eta) > T^*(\eta)$ on $(\eta_+, \infty)$.

In (iii), there exists $\hat{\eta}(+) \in (\eta_+, \infty)$ such that

$T(\eta) > T^*(\eta)$ in $(\hat{\eta}(+), \infty)$,

$T(\hat{\eta}(+)) = T^*(\hat{\eta}(+))$,

and

$T(\eta) < T^*(\eta)$ in $(\eta_+, \hat{\eta}(+))$.

Proof. By (2.18) and (2.24), $T^*_{c}(+) < T_c(+) \iff

\frac{4\alpha(\alpha + \mu^*\beta)}{1 - \mu^*} < \frac{2\alpha(\alpha + \mu^*\beta)}{1 - \mu^*} - \alpha^2$.

By (2.17) and a straightforward computation, the last inequality is shown to be equivalent to $T_+ < Q_*$. Similarly, (ii) and (iii) hold. It remains to show that $T$ and $T^*$ intersect at most once in $(\eta_+, \infty)$. Using (2.22) and (2.23), we obtain

$p_\tau - p^*_{\tau^*} = \frac{\sigma^2 G(p)}{Z(p)(p - p_+)(\tau_+ - \tau)}$,

where

$Z(p) \equiv p + \mu^2 p_+ > 0$,

and

$G(p) \equiv (p - p_+)\{1 - 2\mu^2\} p - p_+ \{1 - 2\mu^2\} p_+ - p\}$

are defined on $(p_+, \infty)$, and $p_+$ is given in (2.6). Since $G$ is quadratic in $(-\infty, \infty)$ and

$G(p_+) = \frac{\sigma^2 \mu^2}{1 - \mu^2}(\mu^2 p_+ - p_+ < 0)$,
$p_\tau > p_A$ implies $G$ has at most one zero in $(p_\tau, \infty)$. Hence, the result follows, and the proof is complete.

When $p_\tau = 0$ or $\tau_\tau = 0$, (2.17) then implies

$$\alpha = \beta = (2\mu^2 Q)^{1/2}.$$  

In this case, by (2.18) and (2.28), we have

$$T_c (0) = T_s (p_\tau, \tau_\tau = 0) = 2\mu^2 Q,$$  

and

$$T_c^\ast (0) = T_s^\ast (p_\tau, \tau_\tau = 0) = \frac{8 - \mu^2}{1 - \mu^2} Q.$$  

On the other hand, when $T_\tau = Q_\ast$, then Theorem 2.7(ii) implies $T_c^\ast (+) = T_s (+)$. In this case, it can be computed that $T_c^\ast (+) = (1 - \mu^4)/(2\mu^2) Q$, allowing us to denote

$$T_c (Q_\ast) = T_s (p_\tau, \tau_\tau = Q_\ast) = \frac{1 - \mu^4}{2\mu^2} Q.$$  

By using (2.32), (2.33) and (2.34) and a straightforward computation, we can obtain

Proposition 2.8.

$$T_c^\ast (0) < T_c (0) < Q_* < T_s (Q_\ast).$$  

As is known from the previous work of Tan and Zhang [8], it is important to determine whether or not the temperatures $T$ and $T^\ast$ along SDT(+) and VN(+) are always higher than the ignition temperature $T_i$. We will show the answer depends on the value relationships among $T_i, T_c (0)$, and $T_s^\ast (0)$.

To begin with we divide all unburnt states $(\tau_\tau, p_\tau)$ into three classes according only to the relationships among $T_i, T_c (0)$, and $T_s^\ast (0)$ as follows.

**Definition 2.9.** If $T_\tau = p_\tau, \tau_\tau < Q_\ast (\gamma)$, then the unburnt state $(\tau_\tau, p_\tau)$ belongs to

- class A if $T_c(0) = T_s^\ast (0)$,
- class B if $T_c(0) < T_s^\ast (0)$,
- class C if $T_c(0) > T_s^\ast (0)$. 

Similarly, if $T_+ > Q_+(\gamma)$, the unburnt state $(\tau_+, p_+)$ belongs to

- class A if $T_i \leq T_+(+)$,
- class B if $T_i \in (T_+(+), T_+^*(+))$,
- class C if $T_i(+)>T_+^*(+)$. 

See Fig. 3, with $T$ and $T^*$ along $SDT(+) \text{ and } VN(+)$. For the fixed ignition temperature $T_i$, denote the set of all unburnt states as

$$U := U(i) = \{(\tau_+, p_+): 0 \leq p_+ \tau_+ < T_i\}$$

Now, we can have a complete classification of all unburnt states as follows.

**Theorem 2.10.** (I) If $T_i \leq Q_+(\gamma)$, then we have three cases:

(i) if $T_i \leq T_+^*(0)$, then each state in $U$ is of class A.

(ii) if $T_+^*(0) < T_i < T_+(0)$, the equi-temperature curve $\Gamma_{AB} \equiv \Gamma_{AB}(i) \equiv \{(\tau_+, p_+) \in U: T_+^*(+) = T_i\}$ divides $U$ into two simply-connected sets, $U_A$ and $U_B$. Each state in $U_X$ is of class $X$ for $X = A$ and $B$.

(iii) if $T_i(0) < T_+$, then there are two disjoint equi-temperature curves $\Gamma_{AB} \equiv \Gamma_{AB}(i) \equiv \{(\tau_+, p_+) \in U: T_+^*(+) = T_i\}$ and $\Gamma_{BC} \equiv \Gamma_{BC}(i) \equiv \{(\tau_+, p_+) \in U: T_+^*(+) = T_i\}$. $\Gamma_{AB}$ and $\Gamma_{BC}$ divide $U$ into three disjoint simply-connected sets $U_A$, $U_B$, and $U_C$. Each state in $U_X$ is of class $X$ for $X = A$, $B$, and $C$.

Similarly, for (II) $T_i > Q_+(\gamma)$, we have two cases:

(i) if $T_i < T_+(Q_+)$, then there exist two disjoint equi-temperature curves $\Gamma_{AB}$ and $\Gamma_{BC}$ defined as in (I)(ii) that divide $U$ into disjoint simply-connected sets $U_A$, $U_B$, and $U_C$. Each state in $U_X$ is of class $X$ for $X = A$, $B$, and $C$. Furthermore, the curve $\{(\tau_+, p_+) \in U: T_+ = Q_+(\gamma)\}$ lies in $U_A$.

(ii) if $T_i > T_+(Q_+)$, then there exist two disjoint equi-temperature curves $\Gamma_{AB} = \{(\tau_+, p_+) \in U: T_+(+) = T_i\}$ and $\Gamma_{BC} = \{(\tau_+, p_+) \in U: T_+^*(+) = T_i\}$ also divide $U$ into three disjoint simply-connected sets $U_A$, $U_B$, and $U_C$. Each state in $U_X$ is of class $X$ for $X = A$, $B$, and $C$.

See Fig. 4.

**Proof.** (I) If $T_i \leq Q_+$ then any unburnt state $(\tau_+, p_+)$ satisfies $T_+ \leq Q_+$. Thus, by Theorem 2.7, we have

$$T_+^*(+) \leq T_i(+)$$

(2.39)
for each $(\tau_+, p_+)\in U$. On the other hand, (2.24) implies
\[ T_{\tau}^*(T_i) = T_{\tau}^*(p_+, \tau_+ = T_i) > T_i. \] (2.40)

(i) If $T_{\tau}^*(0) \geq T_i$, then $T_{\tau}^*(+) > T_{\tau}^*(0)$ implies $T_{\tau}^*(+) > T_i$. Hence each state of $U$ is of class $A$.

(ii) If $T_{\tau}^*(0) < T_i$ then the equi-temperature curve $\Gamma_{AB} = \{ (\tau_+, p_+) \in U : T_{\tau}^*(+) = T_i \}$ is non-empty. Furthermore, (2.40) implies $\Gamma_{AB} \subset U$. It is easy to verify that $\Gamma_{AB}$ is an unbounded, continuous Jordan curve in $U$. Therefore, $\Gamma_{AB}$ divides $U$ into two disjoint open sets,
(II) $T_+ > Q_*(\gamma)$

$$U_A = \{ (\tau_+, p_+) \in U : T^*_{\sigma}(+) < T_i \}$$ and
$$U_B = \{ (\tau_+, p_+) \in U : T^*_{\sigma}(+) > T_i \}$$

(2.39) now implies $T_{\sigma}(+) > T_i$ too. Hence, each state in $U_X$ is of class $X$ for $X = A$ and $B$.

(iii) If $T_{\sigma}(0) < T_i$, then $\Gamma_{AB}$ and $\Gamma_{BC}$ are non-empty in $U$. Now

$$U_A = \{ (\tau_+, p_+) \in U : T^*_{\sigma}(i) > T_i \}$$
$$U_B = \{ (\tau_+, p_+) \in U : T^*_{\sigma}(+) < T_i < T_{\sigma}(+) \}$$
$$U_C = \{ (\tau_+, p_+) \in U : T_{\sigma}(+) < T_i \}$$

It is clear that each state in $U_X$ is of class $X$ for $X = A$, $B$ and $C$. 
(I) $T_i < Q_\gamma(\gamma)$

(II) If $Q_\gamma(\gamma) < T_i$, then the curve $\Gamma_\gamma = \{(\tau_+, p_+) \in U : T_+ = Q_\gamma(\gamma)\} \subset U$.

(i) If $T_i \leq T_c(Q_\gamma)$, then each state in $U^* = \{(\tau_+, p_+) \in U : T_+ > Q_\gamma(\gamma)\}$ is of class $A$ by virtue of Theorems 2.7(iii) and (2.18). Hence, it remains to determine the states in $U_* = \{(\tau_+, p_+) \in U : T_+ < Q_\gamma(\gamma)\}$. 

Fig. 4. Classification of unburnt states.
(II) $T_i > Q_\ast(\gamma)$

Again by Theorem 2.7(i), each state in $U_\ast$ satisfies

$$T_\ast^\gamma(+) < T_i(+)\text{.}$$

Proposition 2.8 and $Q_\ast < T_i$ imply $\Gamma_{AB}$ and $\Gamma_{BC}$ are not empty in $U_\ast$.

A similar argument as in (I) (iii) also holds in $U_\ast$. Hence, the result follows.

(ii) If $T_i > T_i(Q_\ast)$, then by Theorem 2.7(iii) and (2.38), each state in

$$U^\ast = \{(\tau_+, p_+) \in U : T_+ > Q_\ast\}$$

satisfies

$$T_\ast^\gamma(+) > T_i(+)\text{.}$$

On $\Gamma_\ast = \{(\tau_+, p_+) \in U : T_+ = Q_\ast\}$ we have $T_\ast^\gamma(+) = T_i(+) = T_i(Q_\ast) < T_i$ implying that there are two curves,

$\Gamma_{AB} = \{(\tau_+, p_+) : T_i(+) = T_i\}$

and

$\Gamma_{BC} = \{(\tau_+, p_+) : T_\ast^\gamma(+) = T_i\}$

in $U^\ast$, dividing $U^\ast$ into three regions:

$$U_A = \{(\tau_+, p_+) \in U^\ast : T_A(+) > T_i\},$$

$$U_B = \{(\tau_+, p_+) \in U^\ast : T_A(+) < T_i < T_\ast^\gamma(+)\},$$

and

$$U_C = \{(\tau_+, p_+) \in U^\ast : T_\ast^\gamma(+) < T_i\}.$$
In each set, the state is of the indicated class. Finally, in
\[ U_\ast = \{ (\tau_\ast, p_\ast) \in U : T_\ast < Q_\ast \}, \]
each state has \( T_\ast(+) < T_i(+) \) and \( T_i(+) < T_i(Q_\ast) < T_i \). Hence each state in \( U_\ast \) is of class \( C \). The results hold by letting
\[ U_C = U_\ast \cup \bar{U}_C \cup \Gamma_\ast. \]
The proof is complete.

For deflagration waves, we also have results similar to those from Theorem 2.7.

**Definition 2.11.** Given \( Q > 0 \) and \( \gamma \in (1, 2) \), denote
\[ Q^* \equiv Q^*(\gamma) = \frac{1 - \mu^4}{2\mu^2} Q. \]
The positive quantity \( Q^* \) plays a role similar to that of \( Q_\ast \) for detonation waves. Indeed, we have the following theorem.

**Theorem 2.12.** Let \( Q \) and \( T_i \) be given. Then,

(i) \( T(CJDF(i (+))) > T_i \) if and only if \( T_i < Q^* \). (2.41)

(ii) \( T(CJDF(i (+))) = T_i \) if and only if \( T_i = Q^* \). (2.42)

(iii) \( T(CJDF(i (+))) < T_i \) if and only if \( T_i > Q^* \). (2.43)

See Fig. 5.

![Fig. 5. Temperature along WDF.](image)
Proof. By Proposition 2.1, we have
\[ T(CJDF(i(+))) - T_i = \alpha(2\alpha - 2\mu^2\beta)/(1 - \mu^4) - \alpha^2. \] (2.44)
Thus the sign of \( T(CJDF(i(+) - T_i \) is the same as the sign of \( Q^* - T_i \).
The proof is complete.

3. SELFSIMILAR SOLUTIONS OF SZND MODEL

In this section, we study the selfsimilar solutions for the SZND-model. Indeed, let \( \xi = x/t \), if solution \((u, p, \tau, q)\) for (SZND) depends only on \( \xi \), then it satisfies the following equations:

\[
\begin{align*}
\xi u' - p' &= 0, \quad (3.1) \\
\xi^2 \tau' + u' &= 0, \quad (3.2) \\
\xi E' - (up)' &= 0, \quad (3.3) \\
\xi q' &= k \varphi(T) q, \quad (3.4)
\end{align*}
\]
where
\[
\varphi(T) = \begin{cases} 
0, & T \leq T_i, \\
1, & T > T_i.
\end{cases} \tag{3.5}
\]
The Riemann data becomes
\[
(u, p, \tau, q)(-\infty) = (u_-, p_-, \tau_-, 0), \quad (3.6)
\]
and
\[
(u, p, \tau, q)(+\infty) = (u_+, p_+, \tau_+, Q), \quad (3.7)
\]
When \( \xi \neq 0 \) and the solution is smooth, then (3.1) \( \sim \) (3.4) can also be expressed as
\[
\begin{align*}
\xi' &= -\frac{\gamma - 1}{\gamma p - \xi^2 \tau} q', \quad (3.8) \\
p' &= -\xi^2 \xi' , \quad (3.9) \\
q' &= \frac{k}{\xi} \varphi(T) q , \quad (3.10)
\end{align*}
\]
where $\gamma p - \xi^2 \tau \neq 0$. If $\gamma p - \xi^2 \tau = 0$, then we require $q' = 0$. For any $\xi_0 \in \mathbb{R}$,
we may supply (3.8) $\sim$ (3.10) with the initial conditions
\[ \tau(\xi_0) = \tau_0 > 0, \quad p(\xi_0) = p_0 > 0, \quad q(\xi_0) = q_0 \geq 0. \] (3.11)

When solution $(u, \rho, \tau, q)(\xi)$ for the SZND-model has a discontinuity at
$\xi = \eta$, it then satisfies the Rankine–Hugoniot condition,
\[
(\text{RH}) : \quad \eta[u] = [p], \quad (3.12) \\
\eta[\tau] = -[u], \quad (3.13) \\
\[ e + q \] + \frac{p_r + p_l}{2} \tau = 0, \quad (3.14)
\]
where $[w] = w_r - w_l$, $w_r = w(\eta + 0)$, and $w_l = w(\eta - 0)$.

By direct computation, we can determine $(u_\ell, p_\ell, \tau_\ell)$ in terms of
$(u_r, p_r, \tau_r)$, $\eta$ and $[q]$ as follows.
\[
p_\ell = \frac{1}{2} \left\{ (1 - \mu^2) (\eta^2 \tau_r + p_r) \right\} + \frac{1}{2} \left\{ \{ (1 - \mu^2) \eta^2 \tau_r - (1 + \mu^2) p_r \}^2 + 8 \mu^2 \eta^2 \delta \right\}^{1/2}, \quad (3.15) \\
\tau_\ell = \tau_r + (p_r - p_l)/\eta^2, \quad (3.16) \\
u_\ell = u_r + \frac{1}{\eta} (p_l - p_r). \quad (3.17)
\]
When $[q] = 0$, (3.15) $\sim$ (3.17) can be simplified to
\[
p_\ell = (1 - \mu^2) \eta^2 \tau_r - \mu^2 p_r, \quad (3.18) \\
\tau_\ell = (1 + \mu^2) p_r/\eta^2 + \mu^2 \tau_r, \quad (3.19) \\
u_\ell = u_r + (1 - \mu^2) \eta \tau_r - (1 + \mu^2) p_r/\eta. \quad (3.20)
\]

Since state $(-)$ is burnt, we may consider the solution $(u, p, \tau, q)$ is also
burnt in $(- \infty, 0)$, i.e. $q(\xi) = 0$ for any $\xi \in (- \infty, 0]$. Then (3.4) is automati-
cally satisfied, and (3.1) $\sim$ (3.3) is the classic, non-combustion equation.
Therefore, for any data at $\xi = 0$,
\[
\tau(0) = \tau_0 > 0, \quad p(0) = p_0 > 0, \quad u(0) = u_0, \quad q(0) = 0. \quad (3.21)
\]

Then (3.1) $\sim$ (3.3), (3.6) and (3.21) yield unique solutions, (see Chapter 3
in [12]). Therefore, the (SZND) Riemann problem is reduced to finding
suitable initial data $(u_0, p_0, \tau_0, 0)$ at $\xi = 0$ such that solutions for
(3.1) $\sim$ (3.4), (3.7) and (3.21) exist in $(0, \infty)$. Hence, it is necessary to study
the initial-value problem (3.8) ~ (3.11) when \( \zeta_0 \geq 0 \). We first study the existence of global smooth (continuous) solutions in \((\zeta_0, \infty)\), and then study solutions with discontinuities at some \( \eta > \zeta_0 \). The solutions for (3.1) ~ (3.4) depend on \( k(>0) \). For simplicity we shall omit the dependence of \( k \) wherever such omission does not cause confusion.

We first have the following simple observation.

**Lemma 3.1.** Let \( q(\zeta) \) be the solution of (3.10) in \((\zeta_0, \zeta_1)\) with \( q(\zeta_0) = q_0 \geq 0 \). Then:

(i) If \( T > T_i \) in \((\zeta_0, \zeta_1)\) and \( \eta \in (\zeta_0, \zeta_1) \), then

\[
q(\zeta) = q(\eta) \left( \frac{\zeta - \eta}{\zeta_1 - \eta} \right) \quad \text{in } (\zeta_0, \zeta_1).
\]  

(ii) If \( T \leq T_i \) in \((\zeta_0, \zeta_1)\), then

\[
q(\zeta) = q_0 \quad \text{in } (\zeta_0, \zeta_1).
\]

However, if \( T(\zeta_0) = T_i \), then we may have a non-unique solution since \( q \) is discontinuous at \( T = T_i \). This is clarified below.

For more efficient study of (3.8) ~ (3.11), it is convenient to divide \((\mathbb{R}^+)^3\) into different regions, then study the equations for each region separately.

**Definition 3.2.** On \((\mathbb{R}^+)^3\), denote as

\[
G = \{(\tau, p, \zeta) : \gamma p - \zeta^2 \tau > 0\}, \quad I_0 = \{(\tau, p, \zeta) : \gamma \tau = T_i\},
\]

\[
H = \{(\tau, p, \zeta) : \gamma p - \zeta^2 \tau < 0\}, \quad I_+ = \{(\tau, p, \zeta) : \gamma \tau > T_i\},
\]

\[
P = \{(\tau, p, \zeta) : \gamma p - \zeta^2 \tau = 0\}, \quad I_- = \{(\tau, p, \zeta) : \gamma \tau < T_i\},
\]

\[
G^+ = G \cap I_+, \text{ etc.}
\]

Therefore, \((\mathbb{R}^+)^3\) consists of 4 open regions, \( G^+, G^-, H^+, H^- \), 4 surfaces, \( P^+, P^-, G^0, H^0 \) and one curve, \( P^0 \).

For simplicity, denote the solution of (3.8) ~ (3.10) as

\[
X(\zeta) = (\tau(\zeta), p(\zeta), \zeta),
\]  

with initial condition

\[
X(\zeta_0) = X_0 \equiv (\tau_0, p_0, \zeta_0).
\]

We have the following demonstration of solution existence and uniqueness for all regions except \( G^0 \) and \( P^0 \).
Proposition 3.3. Assume $k > 2$. We then have:

(I) (i) If $X_0 \in H^-$, then $X(\xi) = X_0$ in $(\xi_0, \infty)$.

(ii) If $X_0 \in H^0$, then $X(\xi) = X_0$ in $(\xi_0, \infty)$.

(iii) If $X_0 \in H^+$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) \in H^+$ in $(\xi_0, \xi_1)$ and $X(\xi_1) \in H^0$.

(II) (i) If $X_0 \in P^-$, then there exists a $\xi_1 > \xi_0$ such that

\begin{align*}
 p(\xi) &= C_1 \xi^{(2\gamma)/(\gamma + 1)} \\
 \tau(\xi) &= C_2 \xi^{(\gamma - 1)/(\gamma + 1)}
\end{align*}

(3.26)

for $\xi \in (\xi_0, \xi_1)$ and $X(\xi) \in P^0$, for some positive constants $C_1$ and $C_2$.

(ii) If $X_0 \in P^+$ and $q_0 > 0$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) \in H^+ in (\xi_0, \xi_1)$.

(III) (i) If $X_0 \in G^-$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) = X_0$ in $(\xi_0, \xi_1)$ and $X(\xi_1) \in P^-$.

(ii) If $X_0 \in G^+$, then there exists $\xi_1 > \xi_0$ such that $X(\xi) \in G^+$ in $(\xi_0, \xi_1)$ and $X(\xi_1) \in G^0$.

Proof. (I) (i) Since $X_0 \in H^-$, $T(\xi_0) < T$, and $\gamma p_0 - \xi_0^2 \tau_0 < 0$, (3.10) and (3.8) imply $q' = 0$ and $\tau' = 0$ for $\xi > \xi_0$. Hence, $X(\xi) = X_0$ for $\xi > \xi_0$.

(ii) The proof is similar to that for (i).

(iii) When $\gamma p - \xi^2 \tau \neq 0$, we then have

\begin{align*}
 \frac{d}{d\xi} T(\xi) &= -\frac{(\gamma - 1)k}{\xi} \phi(T) \frac{p - \xi^2 \tau}{\gamma p - \xi^2 \tau} \cdot q. \tag{3.27}
\end{align*}

If $X(\xi) \in H^+$, then (3.27) implies $T'(\xi) < 0$. Furthermore, we also have

\begin{align*}
 \frac{d}{d\xi} (\gamma p - \xi^2 \tau) &= -2\xi \tau - (\gamma + 1) \xi^2 \tau' < 0
\end{align*}

in $H^+$. Therefore, $\xi_1 > \xi_0$ exists such that $T(\xi_1) = T$, and $X(\xi) \in H^+$ in $(\xi_0, \xi_1)$.

(II) (i) If $X(\xi) \in P^-$, we then have

\begin{align*}
 \gamma p - \xi^2 \tau &= 0. \tag{3.28}
\end{align*}

For polytropic gas we also have

\begin{align*}
 \rho \tau^2 &= C > 0. \tag{3.29}
\end{align*}
From (3.28) and (3.29), (3.26) follows. Since
\[ T(\xi) = C_0 \xi \eta^{2(k-1)(\gamma+1)}, \] (3.30)

\( T(\xi_1) = T \), for some \( \xi_1 > \xi_0 \). Hence, \( X(\xi_1) \in P^0 \) and \( X(\xi) \in P^- \) in \( (\xi_0, \xi_1) \).

(ii) Since \( X_0 \in P^+ \), \( \xi_0 > 0 \). If there exists \( \xi_1 > \xi_0 \) such that \( X(\xi) \in P^+ \) in \( (\xi_0, \xi_1) \), i.e., (3.28) holds, then (3.30) implies \( T(\xi) > T \). \( q_0 > 0 \) now implies \( q' > 0 \) in \( (\xi_0, \xi_1) \), a contradiction, therefore the result holds.

(III) (i) Since \( X_0 \in G^- \), \( \phi(T) = 0 \), thus \( X(\xi) = X_0 \) in \( (\xi_0, \xi_1) \).

(ii) If \( X(\xi) \in G^+ \), then \( \tau' < 0 \). From (4.11) on p. 344 in [8], we have
\[
\frac{1}{2}(\gamma + 1) \xi^2 \tau^2(\xi) - \frac{1}{2} \xi^2 \tau(\xi) + \gamma p_0 \tau_0 + (\gamma - 1) \xi \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} = 0. \] (3.31)

We may assume \( q_0 > 0 \) in (3.31). Otherwise, we replace \( (\tau_0, p_0, q_0) \) and \( \xi_0 \) with \( (\tau(\xi), p(\xi), q(\xi)) \) and \( \xi \) in (3.31) for some \( \xi > \xi_0 \), and closed to it with \( q(\xi) > 0 \). Since \( k > 2 \), it can be verified by using (3.31) that \( X(\xi) \) can not stay at \( G^+ \) forever. Thus, there exists \( \xi_1 > \xi_0 \) such that \( X(\xi) \in G^+ \) in \( (\xi_0, \xi_1) \), and either \( X(\xi_1) \in G^0 \), or \( X(\xi_1) \in P^+ \). The latter case can be ruled out according to Theorem 3.1 [8]. The proof is complete.

Now it remains to study the problem of \( X_0 \in P^0 \) or \( G^0 \).

**Proposition 3.4.** If \( X_0 \in P^0 \), then either

(i) \( X(\xi) = X_0 \) for \( \xi > \xi_0 \), and so, \( X(\xi) \) stays at \( H^0 \) forever, or

(ii) \( X(\xi) \in G^+ \) for \( \xi \in (\xi_0, \xi_1) \) some \( \xi_1 > \xi_0 \). In this case, we have
\[
\begin{align*}
\tau(\xi) &= \tau_0 - C_0 \xi - \xi_0^{1/2} + o(\xi - \xi_0^{1/2}), \\
p(\xi) &= p_0 + C_0 \xi - \xi_0^{1/2} + o(\xi - \xi_0^{1/2}), \\
T(\xi) &= T + C_0 \xi - \xi_0^{1/2} + o(\xi - \xi_0^{1/2}),
\end{align*}
\] (3.32) (3.33) (3.34)

for \( \xi \sim \xi_0^+ \), where
\[ C_0 = \frac{2(\gamma - 1) k q_0}{(\gamma + 1) \xi_0^{1/2}}. \]

**Proof.** It can be verified that (3.32) and (3.33) hold for \( \xi \sim \xi_0^+ \).
\( X(\xi) \in G^+ \) is equivalent to \( T(\xi) > T \), which is guaranteed by (3.34). The proof is complete.
Similarly, we have a result for $X_0 \in G^0$.

**Proposition 3.5.** If $X_0 \in G^0$, then either

(i) $X(\zeta) = X_0$ in $(\zeta_0, \zeta_1)$ with $X(\zeta_1) \in P^0$, or

(ii) $X(\zeta) \in G^+$ in $(\zeta_0, \zeta_1)$ for some $\zeta_1 > \zeta_0$, and (3.32) $\sim$ (3.34) hold.

**Proof.** If (i) does not hold, then by an argument similar to that in Proposition 3.2(III)(ii) we can prove (ii), the details are omitted. The proof is complete.

The case of $\zeta_0 = 0$ is the most interesting to us, since we have to solve the Riemann problem in $(0, \infty)$. When $\zeta_0 = 0$, we always assume (3.21), and then $X_0 \in G^+ \cup G^0$. If $X(\zeta) \in G^+$, for $\zeta > 0$, then (3.10) implies

$$q(\zeta) = \tilde{q}_0 \zeta^k$$

for $\zeta \sim 0^+$, where $\tilde{q}_0 \geq 0$ is a parameter. In view of (3.35), we always have the freedom to choose $\tilde{q}_0 \in [0, \infty)$ and it may be possible to find an appropriate $\tilde{q}_0$ to fit the boundary conditions at some point $\eta (\leq \infty)$ when it is needed. This is stated more precisely below. The following proposition is very important in studying temperature of $X(\zeta)$ at $G^+$, which was essentially proven in [8].

**Proposition 3.6.** If $\zeta_0 > 0$, $X(\zeta_0) \in G^+$ and

$$T(\zeta_0) = 0,$$  

then there exists $\zeta_1 > \zeta_0$ such that $X(\zeta) \in G^+$

$$T(\zeta) < 0 \quad \text{in} \quad (\zeta_0, \zeta_1) \quad \text{and} \quad T(\zeta_1) = T_i,$$  

i.e., $X(\zeta_1) \in G^0$.

**Proof.** By Theorem 3.2 in [8], there is no $\zeta > \zeta_0$ such that $X(\zeta) \in G^+$ and $T(\zeta) = 0$. Therefore, $T(\zeta) < 0$ as far as $X(\zeta) \in G^+$. Now, according to Proposition 3.2 (III)(ii), there exists $\zeta_1 > \zeta_0$ such that $T(\zeta_1) = T_i$. The proof is complete.

**Remark 3.7.** If $\zeta_0 = 0$ and $X_0 \in G^+$, then $T(0) = 0$, which does not contradict Proposition 3.5. Furthermore, according to Proposition 3.5, the temperature $T(\zeta)$ either strictly decreases or has exactly one maximum in $(0, \zeta_1)$ with $X(\zeta_1) \in G^0$, where $\zeta_1$ is the first $\zeta$ such that $T(\zeta) = T_i$. Summarizing the above results, we have the following diagram for constructing the solutions starting at $\zeta_0 \geq 0$, see Fig. 6.
Combining the results of Propositions 3.3, 3.4 and 3.5, we have the following global existence of solutions for (3.8) - (3.10).

**Theorem 3.8.** Given \( \tau(0) > 0, \ p(0) > 0 \) and \( q(0) = 0 \), then there is a smooth (continuous) solution \( X(\xi) \) for (3.8) - (3.10) in \((0, \infty)\). Furthermore, there is \( \xi \leq \infty \) such that \( q(\xi) \leq Q \) in \((0, \xi)\) and \( q(\xi) = Q \) if \( \xi < \infty \).

Due to the non-uniqueness of solutions when \( X_0 \in P^0 \cup G^0 \), we shall pay much more attention to the following simplest solutions which were considered in [8].

**Definition 3.9.** A solution \( X(\xi) \) for (3.8) - (3.10) is called simple

(i) if \( X(0) \in G^0 \cup G^+ \), then \( X(\xi) \) can not jump to \( G^+ \) in \((0, \xi)\),

(ii) if \( X(0) \in G^- \), then \( X(\xi) \) jumps exactly once in \((0, \xi)\).

Otherwise, \( X(\xi) \) is called a non-simple solution.

Therefore, we have four types of simple solutions when \( q(\xi) \leq Q \); see Fig. 7. From (3.10), because \( q(\xi) \) never decreases, \( X(\xi) \) is a physical solution of (SZND), and it is necessary that \( q(\xi) \leq Q \). It is possible to get a non-simple solution while \( q \leq Q \).

For type (i) solutions, in [8] Tan and Zhang considered \( q(\eta_0) = Q \) for some \( \eta_0 > 0 \). (3.8) - (3.10) are then equivalent to

\[
\tau' = - \frac{k(\gamma - 1)}{\gamma p - \xi^2 \tau} \left( \frac{\xi}{\eta_0} \right)^k - 1, \quad (3.38)
\]

\[
p' = - \xi^2 \tau', \quad (3.39)
\]
where

\[ q(\xi) = Q \left( \frac{\xi}{\eta_0} \right)^k. \]  

(3.40)

Based on (3.38) \sim (3.40), Tan and Zhang were able to prove their results, the details are presented in the next section.

4. LIMITS OF SZND

In this section, we shall study the limits of selfsimilar simple solutions of (SZND) and the locate the states in \( J(+) \) which can be limits of (SZND)
as $k \to \infty$. The methods of proof are similar to those used in [8], here we only sketch a necessary modification and omit other details.

Let

$$\tau_k(0) = \tau_{\infty} > 0, \quad p_k(0) = p_{\infty} > 0, \quad q_k(0) = 0,$$

(4.1)

and $X_k(\xi) = (\tau_k(\xi), p_k(\xi), \xi)$ be the solutions of (3.8) ~ (3.10). Let $\eta_k < \infty$ such that

$$q_k(\eta_k) = Q.$$

(4.2)

If $X_k(\xi) \in G^+$ in $(0, \eta_k)$, i.e.,

$$T_k(\xi) > T, \quad \text{in} \ (0, \eta_k),$$

(4.3)

Then

$$q_k(\xi) = Q \left( \frac{\xi}{\eta_k} \right)^k,$$

(4.4)

and $X_k(\xi)$ satisfies

$$\tau_k^\prime(\xi) = \frac{kQ(\gamma - 1)}{jp_k - \xi^2\eta_k} \left( \frac{\xi}{\eta_k} \right)^{k-1} \frac{1}{\eta_k},$$

(4.5)

$$p_k^\prime(\xi) = \frac{kQ(\gamma - 1)}{jp_k - \xi^2\eta_k} \eta_k \left( \frac{\xi}{\eta_k} \right)^{k+1},$$

(4.6)

and

$$u_k^\prime = \frac{-kQ(\gamma - 1)}{jp_k - \xi^2\eta_k} \left( \frac{\xi}{\eta_k} \right)^k.$$

(4.7)

Condition (4.3) implies the solution $X(\xi)$ under consideration is necessarily of type (i) or (ii). For type (iii) or (iv), (4.3) may be replaced by

$$T_k(\xi) > T, \quad \text{in} \ (\hat{\eta}_k, \eta_k),$$

with $T_k(\hat{\eta}_k) = T$. All results obtained below for type (i) or (ii) can also be applied to type (iii) or (iv) on $(\hat{\eta}_k, \eta_k)$. Therefore, we only consider type (i) or (ii) solutions.
The problem of studying the limits of (4.4)–(4.7) as $k \to \infty$ is necessarily a singularity problem. For example, for type (i) or (ii) solutions, if

$$\eta_k \to \eta_0 (\leq \infty) \quad \text{as} \quad k \to \infty,$$

then (4.4) implies

$$q_k(\xi) \to q^*(\xi) = 0 \quad \text{in} \quad (0, \eta_0) \quad \text{as} \quad k \to \infty,$$ (4.8)

and

$$q_k(\eta_0) = Q = q^*(\eta_0) \quad \text{for all} \quad k.$$

Hence, the limit $q^*$ is 0 in $(0, \eta_0)$ and $Q$ at $\eta_0$, a jump at $\eta_0$. This phenomena is also brought to the limit $(\tau^*, p^*)$ of $(\tau_k, p_k)$ as $k \to \infty$, whenever $(\tau^*, p^*)$ exists. This is discussed in more detail below. We first state a monotonicity result for $\tau_k$ and $T_k$ in $k$ when $X_k \in G^+$. 

Proposition 4.1. Assume (4.1), (4.2), (4.3) and

$$(\tau_{0k}, p_{0k}, \eta_k) \to (\tau_0, p_0, \eta_0) \quad \text{as} \quad k \to \infty.$$ (4.9)

Then for sufficient large $k$, the solution $X_k$ satisfies

(i) $\frac{\partial \tau_k}{\partial k} \geq 0$ \quad in \quad $(0, \eta_0),$

(ii) $\frac{\partial T_k}{\partial k} \geq 0$ \quad in \quad $(\bar{\eta}, \eta_0)$ \quad for \quad some \quad $0 < \bar{\eta} < \eta_0$.

Proof. (i) This can be proven by the same argument used in [8] since (4.9) holds, the detail is omitted.

(ii) By (3.9), we have

$$p_k(\xi) = -\xi^2 \tau_k(\xi) + 2 \int_{\zeta}^{\xi} s \tau_k(s) \, ds.$$ 

Now,

$$\frac{\partial p_k}{\partial k}(\xi) = -\xi^2 \frac{\partial \tau_k}{\partial k}(\xi) + 2 \int_{\zeta}^{\xi} s \frac{\partial \tau_k}{\partial k}(s) \, ds.$$
implies
\[ \frac{\partial T_k}{\partial k}(\zeta) = (p_k(\zeta) - \zeta^2 \tau_k(\zeta)) \frac{\partial \tau_k}{\partial k}(\zeta) + 2 \tau_k \right|_{s=0} \int_s^\zeta \frac{\partial \tau_k}{\partial k}(s) \, ds. \]

Then (ii) follows by (i) and \( X_k \in G^+ \). The proof is complete.

The above theorem enables us to define the limits of solutions \( X_k \).

**Definition 4.2.** Let (4.1), (4.2), (4.3) and (4.9) hold, and \( X_k \) by type (i), define
\[
\begin{align*}
p^*(k) &= \lim_{k \to \infty} p_k(\zeta), \\
\tau^*(k) &= \lim_{k \to \infty} \tau_k(\zeta), \quad \text{in } (0, \eta_0), \\
p^*(\eta_0) &= \lim_{k \to \infty} p_k(\eta_0) \quad \text{and} \quad \tau^*(\eta_0) = \lim_{k \to \infty} \tau_k(\eta_0).
\end{align*}
\]

For type (ii) solution, we also assume \( T(\eta_0) = T^* \).

**Proposition 4.3.** Assume (4.1), (4.2), (4.3) and (4.9) hold.

(i) If \( \eta_0^2 \leq \gamma p_0/\gamma_0 \), then \((\tau^*, p^*) = (\tau_0, p_0) \) in \((0, \eta_0)\) and the convergence of \((\tau_k, p_k)\) to \((\tau^*, p^*)\) is uniformly in any compact subinterval of \([0, \eta_0]\).

(ii) If \( \eta_0^2 > \gamma p_0/\gamma_0 \), and \( \tilde{\eta}_0 = (\gamma p_0/\gamma_0)^{1/2} \), then \((\tau^*, p^*) = (\tau_0, p_0) \) in \([0, \tilde{\eta}_0]\) and a rarefaction wave given in (3.26) in \([\tilde{\eta}_0, \eta_0]\).

**Proof.** The results can be obtained by an argument similar to that used in [8], so details are omitted.

The main part of the following theorem was essentially proven in [8].

**Theorem 4.4.(I)** If \((\tau_0, p_0) \in SDT(+) \) with \(-\eta_0^2 = (p_0 - p_+) / (\tau_0 - \tau_+)\), then
\[
T^*(\eta_0) \geq T^*, \quad T(\eta_0) \geq T^*, \quad \tau^*(\eta_0) = \tau^*(\eta_0) \quad \text{and} \quad p^*(\eta_0) = p^*(\eta_0)
\]
(4.10)
hold if and only if
\[
\lim_{k \to \infty} (\tau_0, p_0, \eta_0) = (\tau_0, p_0, \eta_0)
\]
(4.11)
such that the associated solution \( X_k \) is of type (i).
(II) If \((\tau_0, p_0) \in R(CJDT(+)\)) with
\[-\eta^2_0 = \frac{p_+ - p_+}{\tau_0 - \tau_+} \quad \text{and} \quad \eta^2_0 > \gamma p_0 / \tau_0.\]

Case 1.
\[T_0 \geq T_t, \quad \tau^*(\eta_0) \geq T_t, \quad \tau^*(\eta_0) = \tau^*(\eta_0) \quad \text{and} \quad p^*(\eta_0) = p^*(\eta_0) \tag{4.13}\]
hold if and if (4.11) holds such that \(X_k\) is of type (i).

Case 2.
\[T_0 < T_t \tag{4.14}\]
and (4.14) hold if and only if (4.11) holds such that \(X_k\) is of type (iii).

(III) If \((\tau_0, p_0) \in WDF(i(+))\) with \(-\eta^2_0 = (p_0 - p_i)/(\tau_0 - \tau_+),\) the
\[-\eta^*(\eta_0) = \tau_+ \quad \text{and} \quad p^*(\eta_0) = p_i \tag{4.15}\]
hold if and only if (4.11) holds such that \(X_k\) is of type (ii).

(IV) If \((\tau_0, p_0) \in R(CJDF(i(+))\) with
\[-(\eta')^2 = \frac{p' - p_i}{\tau' - \tau_+} \quad \text{and} \quad (\eta')^2 > \gamma p_0 / \tau_0.\]

Case 1.
\[T_0 \geq T_t, \quad \tau^*(\eta') = \tau_+ \quad \text{and} \quad p^*(\eta') = p_i \tag{4.16}\]
hold if and only if (4.11) holds such that \(X_k\) is of type (ii).

Case 2. (4.14) and (4.16) hold if and only if (4.11) holds such that \(X_k\) is of type (iv).

Proof. The proofs of (I), (II), and (III) are almost the same as those given in [8]. However, \(\eta_k\) is not necessarily equal to \(\eta_0,\) and \(q(\eta_k)\) may be less than \(Q\) in general, which is assumed in [8]. In proving (IV), we need the following result:

Suppose \((\tau', p')\) is the CJDF(i(+)) and \((\eta')^2 = -(p' - p_+)/(\tau' - \tau_+).\) Then (4.16) holds.

Indeed, by Lemma 2.1, we have \((\eta')^2 = (\beta - \alpha)/(1 - \mu^2) \tau_+.\) Then, from [8], we have \(p^*(\eta') = p' + \eta' \alpha\) and \(\tau^*(\eta') = \tau' - \alpha / \eta'.\) A direct computation implies (4.16). The proof is complete.
Now, combining the classification theorems in Section 2 and Theorem 4.4, we obtain the following complete result for $J(+)$. 

**Theorem 4.5.** (I) Concerning $JDT(+)$:

- (i) If unburnt state $(+)$ is of class $A$, then any state $(\tau_0, p_0) \in JDT(+)$ is a limit of the simple solution $X_k$ of (SZND) as $k \to \infty$. Furthermore, $X_k$ is of type (i) when $T_0 \geq T_i$, and of type (iii) when $T_0 < T_i$.

- (ii) If unburnt state $(+)$ is of class $B$ or $C$, then state $(\tau_0, p_0) \in SDT(+) \text{ is a limit of the simple solution } X_k \text{ of (SZND) as } k \to \infty \text{ if and only if}$

$$\eta_0 \geq \eta^*, \quad (4.17)$$

where

$$T^*(\eta^*) = T_i. \quad (4.18)$$

In this case, $X_k$ is of type (i). Furthermore, any state in $R(CJDT(+))$ cannot be a limit of the simple solution of (SZND).

(II) Concerning $JDF(+)$:

- (i) If $Q^*(\gamma) \geq T_i$, then for any unburnt state $(+)$, any state $(\tau_0, p_0) \in JDF(+)$ can be a limit of the simple solution $X_k$ of (SZND) as $k \to \infty$. Furthermore, $X_k$ is of type (ii) when $T_0 \geq T_i$, and of type (iv) when $T_0 < T_i$.

- (ii) If $Q^*(\gamma) < T_i$ and $(+)$ is an unburnt state, then $(\tau_0, p_0) \in WDF(i(+)) \text{ is a limit of the simple solution } X_k \text{ of (SZND) as } k \to \infty \text{ if and only if}$

$$\eta_0 \geq \eta^*, \quad (4.19)$$

where

$$T(\eta^*) = T_i. \quad (4.20)$$

In this case, $X_k$ is of type (ii). Furthermore, any state in $R(CJDF(i(+)))$ cannot be a limit of the simple solution of (SZND).

**Remark 4.6.** It is of interest to study the limit of non-simple solutions of (SZND) as $k \to \infty$.

**ACKNOWLEDGMENTS**

We thank Professor T. Zhang for some stimulating discussions related to this work when he was visiting our department in March 1995.
REFERENCES


