Lovász Local Lemma (I)

Lecturer: Dr. Hong-Gwa Yeh
Department of Mathematics
National Central University
Taiwan
hgyeh@math.ncu.edu.tw
**Dependency Digraph**

**Def:** A dependency digraph for a set of events $A_1, \ldots, A_n$ is a digraph $\overrightarrow{G} = (V, E)$ s.t. $V = \{1, 2, \ldots, n\}$ and, for each $i$, $1 \leq i \leq n$ the event $A_i$ is mutually independent of the events $\{A_j : i \neq j\}$.

**Note:**

1. Events $E_1, \ldots, E_n$ are mutually independent iff for any subset $I \subseteq [n]$
   \[ \Pr\left( \bigcap_{i \in I} E_i \right) = \prod_{i \in I} \Pr(E_i) \]

2*. We say that an event $E$ is mutually independent of the events $E_1, \ldots, E_n$ if for any subset $I \subseteq [n]$, \[ \Pr(E \cap \bigcap_{j \in I} E_j) = \Pr(E) \]
Lovász Local Lemma: general case

Thm (LLL, General Case)
Let $D = (V, E)$ be a dependency digraph of events $A_1, A_2, \ldots, A_n$.
If $\exists 0 \leq x_i < 1, \text{ s.t. } P(A_i) \leq x_i \prod_{j \in N^+(i)} (1-x_j), \forall i$
then $P(\bigcap_{i=1}^n \overline{A_i}) = \prod_{i=1}^n (1-x_i) > 0$

pf: Claim For $i \notin S \subseteq \{1, 2, \ldots, n\}$, $P(A_i | \bigcap_{j \in S} \overline{A_j}) \leq x_i$

pf: By induction on $|S|$. Let $B = \bigcap_{j \in S_1} \overline{A_j}$, $C = \bigcap_{j \in S_2} \overline{A_j}$ where $S_1 = S \cap N^+(i) \overset{\text{def}}{=} \{A_1, \ldots, A_k\}$, $S_2 = S \setminus S_1$

$P(B | C) = P(\overline{A_{k_1}} | \overline{A_{k_2}} \cap \ldots \cap \overline{A_{k_r}} \cap C)$
$P(\overline{A_{k_2}} | \overline{A_{k_3}} \cap \ldots \cap \overline{A_{k_r}} \cap C)$
$\vdots$
$P(\overline{A_{k_r}} | \overline{A_r} \cap C)$
$P(\overline{A_r} | C)$
$\geq (1-x_{k_1})(1-x_{k_2}) \ldots (1-x_{k_r})$ by induction hypothesis
\[
P(A_i \cap \overline{A}_j) = P(A_i \mid B \cap C) = \frac{P(A_i \cap B \cap C)}{P(B \cap C)} = \frac{P(A_i \cap B \cap C)}{P(B \cap C)} \leq \frac{P(A_i \mid B \cap C)}{P(B \mid C)} \cdot \frac{P(A_i)}{P(B \mid C)} \leq \frac{x_i \prod_{j \in N \cap (i)} (1 - x_j)}{(1 - x_0) \ldots (1 - x_r)} \leq x_i
\]

Therefore

\[
P(\cap_{i=1}^{n} \overline{A}_i) = P(\overline{A}_i \cap \overline{A}_{i+1}) = P(\overline{A}_i \cap \overline{A}_{i+2}) = P(\overline{A}_i \cap \overline{A}_{i+3}) = P(\overline{A}_i \cap \overline{A}_{i+4}) = \cdots = P(\overline{A}_i \cap \overline{A}_n)
\]

\[
P(\overline{A}_i) = \prod_{i=1}^{n} (1 - x_i) > 0
\]

QED of Claim
Lovász Local Lemma: symmetric case

**Thm (LLL, Symmetric case)**

Let $D$ be a dependency digraph of events $A_1, \ldots, A_n$ with $\text{deg}(i) \leq d \forall i$.

If $P(A_i) \leq \frac{1}{e(d+1)} \forall i$, then $P(\bigcap_{i=1}^n \overline{A_i}) > 0$

**Pf:** For $d \neq 0$, set $x_i = \frac{1}{d+1} < 1$.

$$P(A_i) \leq \frac{1}{e(d+1)}$$

$$\leq \frac{1}{(1 + \frac{1}{d})^d (d+1)}$$

$$= \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d$$

$$\leq x_i \prod_{j \in N(i)} (1 - x_j)$$

$y = \frac{1}{x}$

$f(x) = \left(1 + \frac{1}{x}\right)^x$

QED
Ramsey Numbers

\[ R(G,H) \overset{\text{def}}{=} \min \{ n : \text{every red-blue edge coloring of } K_n \text{ contains a red } G \text{ or a blue } H \} \]

\[ R(k,l) \overset{\text{def}}{=} R(K_k, K_l) \]

Facts 1. \( R(K_1, G) = 1 \)
2. \( R(K_2, G) = 1 + |V(G)| \)

Thm \( V_G, V_H \geq 2. \) Then \( R(G,H) \leq R(G',H) + R(G,H') \). Moreover if both \( R(G',H) \) & \( R(G,H') \) are even then \( R(G,H) < R(G',H) + R(G,H') \).

Def: let \( n = R(G',H) + R(G,H') \).

(1) Red-blue edge color \( K_n \) to get
\[
\{ \geq R(G',H) \} \quad \text{or} \quad \{ = R(G',H') \}
\]
Thus \( R(G,H) \leq n \).

(2) Assume \( R(G,H) = n \). So \( \exists \) a red-blue edge coloring of \( K_{n-1} \) s.t. no red \( G \) & no blue \( H \). For every vertex \( v \) in \( K_n \)
\[
\{ < R(G',H) \} \quad \text{and} \quad \{ < R(G,H') \}
\]
Thus \( R(G',H) = R(G,H') = R(G',H) - 1 \).

And \[ |\text{red edges}| = \frac{(n-1)(R(G',H)-1)}{2} \] a contradiction since \( n \) and \( R(G',H) \) are even.

QED
Bounding $R(k,k)$ with LLL

**Thm**

If $e \in \{ \binom{k}{2} \binom{n}{k-2} + 1 \} 2 \left( \frac{1}{2} \right) \binom{k}{2} \leq 1$ then $R(k,k) > n$

**pf:**

For $s \in [n]$, let $A_s = \{ G_{n,\frac{1}{2}}[s] \cong K_k \text{ or } \overline{K_k} \}$.

Note that $\Pr(A_s) \leq 2 \Pr(G_{n,\frac{1}{2}}[s] \cong K_k) = 2 \left( \frac{1}{2} \right) \binom{k}{2}$

and $\forall \{ t \in [n] : |T \cap S| \geq 2 \} \leq \binom{k}{2} \binom{n}{k-2}$.

$\Pr(A_s) \in \{ \Delta + 1 \} \leq 1$

Maximum outdegree of the dependency digraph on events $A_s$.

$\Rightarrow \Pr(\bigcap_{s \in [n]} \overline{A_s}) > 0$

$\Rightarrow R(k,k) > n$

QED
Corollary \[ R(k,k) > \frac{\sqrt{2}}{e} k 2^\frac{k}{2} (1 + o(1)) \]

**pf:** \[ e \{ \left( \binom{k}{2} \binom{n}{k-2} + 1 \right) 2^{\left( \frac{1}{2} \right)^\binom{k}{2}} \leq 1 \]
\[ \iff 3 \left( \frac{k}{2} \right) 2^{l-\binom{k}{2}} \leq 1 \text{ as } n \to \infty \]
\[ \iff 3 \left( \frac{k}{2} \right) \frac{k(k-1)}{(n-k+2)(n-k+1)} \binom{n}{k} 2^{1-\binom{k}{2}} \leq 1 \]
\[ \iff 3 \frac{\frac{k^2}{(n-k)^2}}{\frac{\sqrt{2\pi k}}{(k/e)^k}} 2^{-\frac{k(k-1)}{2}} \leq 1 \]
\[ \iff 3 \frac{\frac{k^2}{(n-k)^2}}{\left( \frac{he}{k} \right)^k} 2^{-\frac{k(k-1)}{2}} \leq 1 \]
\[ \iff 3 \frac{\frac{k^2}{(n-k)^2}}{\left( \frac{he}{k} \right)^k} \left( \frac{he}{k} \right)^k 2^{-\frac{k(k-1)}{2}} \leq 1 \]
\[ \iff 3 \frac{\frac{k^2}{(n-k)^2}}{\left( \frac{he}{k} \right)^k} \left( \frac{he}{k} \right)^k 2^{-\frac{k(k-1)}{2}} \leq 1 \]
\[ \iff 3 \frac{k^2}{k^2 \left( \frac{\sqrt{2}}{e} 2^{k} (1-\epsilon) - 1 \right) \leq 1}, \text{ here } n \sim \frac{\sqrt{2}}{e} k 2^\frac{k}{2} (1-\epsilon) \]
\[ \iff 3 \frac{\frac{k^2}{e^2 2^k (1-\epsilon)^2}}{2^k (1-\epsilon)^k} \leq 1, \text{ as } k \to \infty \]
\[ \iff \frac{3 e^2}{2} k^2 (1-\epsilon)^{k-2} \leq 1, \text{ as } k \to \infty \]

QED
Bounding $R(k,k)$ with alternation method

**Theorem.** For $n \in \mathbb{Z}^*$ and $p \in [0,1]$, $R(k,k) > n - \binom{n}{k} p^k (1-p)^{n-k}$

**Proof:** To show $\exists$ a graph of order $n$ with at most $\binom{n}{k} p^k (1-p)^{n-k}$ induced $K_k$ and $\binom{n}{k} (1-p)^{n-k}$ induced $\overline{K_k}$. Deleting a single vertex in $K_k$ (resp. $\overline{K_k}$) will destroy $K_k$ (resp. $\overline{K_k}$). Consider G.m.p. $X_k(w) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } w[R] \equiv K_k \\ 0 & \text{o.w.} \end{cases}$, $X_k(w) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } w[B] \equiv K_k \\ 0 & \text{o.w.} \end{cases}$

$Z \overset{\text{def}}{=} n - \sum_{K_k \in [k]} X_k - \sum_{B_k \in [k]} X_k \cdot \mathbb{E}Z = n - \binom{n}{k} p^k (1-p)^{n-k}$

**Corollary.** $R(k,k) \geq \frac{k}{e (1+o(1))} 2^{\frac{k^2}{2}}$

**Proof:** $R(k,k) > n - 2 \binom{n}{k} 2^{-\binom{k}{2}} > n - \frac{1}{2^{2k}} \left( \frac{e^2}{4} \right)^k 2^{-\binom{k}{2}} = n - \frac{2}{\sqrt{2\pi k}} \left( \frac{e\pi}{k} \right)^k 2^{-\binom{k}{2}}$

$> n - \left( \frac{e\pi}{k} \right)^k 2^{-\frac{k^2}{2}} = n - \left[ 2^{\frac{k}{2}} \left( 1+o(1) \right) \right]^k 2^{-\frac{k^2}{2}} \left( \text{set } n \sim \frac{e}{e (1+o(1))} 2^{\frac{k^2}{2}} \right)$

$\sim \frac{k}{e} (1+o(1)) 2^{\frac{k^2}{2}}$

**QED**
Bounding \( R(3,k) \) with LLL

**Observation** \( R(3,k) > c \left( \frac{k}{\ln k} \right)^2 \)

**pf:** sample space = \( \mathbb{G}_{n,p} \).

For \( T \in \{3\} \), \( S \in \{k\} \), let \( A_T = \{ G_{n,p}[T] \cong K_3 \} \), \( B_S = \{ G_{n,p}[S] \cong K_k \} \).

Consider dependency digraph of the events \( A_T \) and \( B_S \):

\[ A_T, \# \leq \binom{3}{2} \binom{n}{1} \leq 3n \]

\[ A_T', \# \leq \binom{k}{2} \binom{n}{1} \leq \frac{k^2 n}{2} \]

\[ B_S, \# \leq \binom{n}{k} \]

\[ B_S', \# \leq \binom{n}{k} \]

**Recall:**

LLL says that, for events \( A_1, \ldots, A_n \) with dependency graph \( D \), if \( \exists x_1, x_2, \ldots, x_n \) such that

1. \( 0 \leq x_i < 1 \), \( i = 1, 2, 3, \ldots, n \)

2. \( \mathbb{P}(A_i) \leq x_i \prod_{j=1}^{i-1} (1-x_j), \quad 1 \leq i \leq n \)

then

\[ \mathbb{P}(\bigcap_{i=1}^{n} \overline{A_i}) > 0 \]
If there exist values $x, y \in [0,1]$ and $p \in (0,1)$ such that

\[
\begin{align*}
    p^3 &\leq x (1-x)^3 n (1-y)^k \binom{n}{k} \\
    (1-p)^{\binom{k}{2}} &\leq y (1-x)^{\binom{k}{2}} (1-y)^{\binom{n}{k}}
\end{align*}
\]

then $P(\bigcap_{T \in \mathcal{E}} \bar{A}_T \bigcap_{S \in \mathcal{P}} \bar{B}_S) > 0$.

That is $R(3, k) > n$. 

$P(A_T)$

$P(B_s)$
What is the largest $k = k(n)$ so that there exist $p, y, x$ satisfying these conditions?

Well, elementary analysis (and a free weekend!)

give that the best choice is achieved at

$p = c_1 n^{-\frac{1}{2}}, \quad K = c_1 n^{\frac{1}{2}} \ln n, \quad x = c_3 n^{-\frac{3}{2}} \quad \text{and} \quad y = \exp\{n^{\frac{1}{2}} \ln^2 n\}$

We have $k = c_1 n^{\frac{1}{2}} \ln n \Rightarrow \frac{k^2}{\ln k} \geq \frac{1}{2} \ln n \Rightarrow n = c \frac{k^2}{\ln^2 k}$

Therefore $R(3, K) > c (\frac{k}{\ln k})^2$

QED
A Story of Joel H. Spencer:

The values \( R(3,3) \) and \( R(4,4) \) were found by Greenwood and Gleason in 1955. As Gleason was my advisor, I once spent a weekend puzzling over \( R(5,5) \) and then asked him for advice. He was quite clear: "Don't work on it!" Behind their elegant paper lay untold hours of calculation in attempts at improvement. The Law of Small Numbers was at work—simple patterns for \( k \) small disappear when \( k \) gets too large for easy calculation. Indeed, in the two decades since that advice, even \( R(4,5) \) has remained a mystery. (There has been more success with \( R(3,k) \) with \( 3 \leq k \leq 9 \) now known.)

Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of \( R(5,5) \) or they will destroy our planet. In that case, he claims, we should marshall all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for \( R(6,6) \). In that case, he believes, we should attempt to destroy the aliens.
Summary

For lower bound of $R(3,k)$:

- Basic probabilistic method gave "Nothing".
- The alteration method gave $R(3,k) > k^{3/2} + o(1)$.
- The LLL gave $R(3,k) > c\left(\frac{k}{\ln k}\right)^2 = k^{2+o(1)}$.
- Kim (1995) gave $R(3,k) > c\frac{k^2}{\ln k}$.

For lower bound of $R(4,k)$:

- Basic probabilistic method gave $R(4,k) > c\left(\frac{k}{\ln k}\right)^{3/2} = k^{3/2+o(1)}$.
- The alteration method gave $R(4,k) > c\left(\frac{k}{\ln k}\right)^2 = k^{2+o(1)}$.
- The LLL gave $R(4,k) > k^{5/2+o(1)}$. 
Lovász Local Lemma: special case

Thm (Lovász's exercise book) Let \( \vec{G} = (V, \vec{E}) \) be a dependency digraph of the events \( A_1, \ldots, A_n \) with \( d^d(i) \leq d \) \( \forall i \).

If \( \Pr(A_i) \leq \frac{1}{4d} \) \( \forall i \), then \( \Pr(\bigwedge_{i=1}^n \overline{A_i}) > 0 \).

PF: First, we prove \( \Pr(A_1 A_2 \cdots A_n) \leq \frac{1}{2d} \) by induction on \( n \). \( N_i := \{ 2, \ldots, n \} \)

\[
\text{LHS} = \frac{\Pr(A_1 A_2 \cdots A_n | \overline{A}_{n+1} \cdots \overline{A}_n)}{\Pr(\overline{A}_2 A_3 \cdots A_n | \overline{A}_{n+1} \cdots \overline{A}_n)} \leq \frac{\Pr(A_1 | \overline{A}_{n+1} \cdots \overline{A}_n)}{1 - \Pr(A_2 A_3 + \cdots + A_n | \overline{A}_{n+1} \cdots \overline{A}_n)} = \frac{\Pr(A_1)}{1 - \Pr(A_2 A_3 + \cdots + A_n | \overline{A}_{n+1} \cdots \overline{A}_n)} \\
\leq \frac{1/4d}{1 - \sum_{i=2}^n \Pr(A_i | \overline{A}_{i+1} \overline{A}_{i+2} \cdots \overline{A}_n)} \leq \frac{1/4d}{1 - (1 - 1/2d)(1 - 1/2d)} \leq \frac{1/4d}{1 - 1/2d} = \frac{1}{2d}.
\]

It turns out that \( \Pr(\bigwedge_{i=1}^n \overline{A_i}) = \Pr(A_1 A_2 \cdots A_n) \Pr(A_2 \cdots A_n) \geq (1 - \frac{1}{2d}) \Pr(A_2 A_3 \cdots A_n) > 0 \)

(this is proved by induction on \( n \), under the induction hypothesis \( \Pr(A_2 A_3 \cdots A_n) > 0 \))

QED
**K-satisfiability problem**

K-CNF: a Boolean formula s.t. each clause has exactly \( k \) distinct literals.

\[
(x_1 \lor \neg x_1 \lor x_2 \lor x_3) \lor (x_3 \lor \neg x_2 \lor \neg x_1 \lor x_4)
\]

\[
(x_1 + \bar{x}_1 + x_2 + x_3) \lor (x_3 + \bar{x}_2 + x_1 + x_4)
\]

**KSAT**: Given a Boolean formula in K-CNF is there a satisfying truth assignment to the input variables?

**Thm**: If no variable in a KSAT formula appears in more than \( \frac{2^k}{4k} \) clauses, then the formula has a satisfying assignment.

**pf**: Consider a random assignment to the variables.

Assume the k-CNF has the form \( x_{out} = V_i \in \mathbb{m} (\bigwedge_{j=1}^k l_{ij}) \).

Let \( E_i \overset{\text{def}}{=} \{ \text{ith clause is NOT satisfied} \} \) \( i = 1, 2, \ldots, m \).

Note that \( E_i \) contains clauses containing \( x_i \) *\( \leq k \cdot \frac{2^k}{4k} = \frac{2^k}{2} = \frac{2^{k-2}}{2} \).

And \( \Pr(E_i) 4d \leq 2^k \cdot 4 \cdot 2^{k-2} = 1 \). The Thm follows from LLL.

QED
2-coloring hypergraph with LLL

**Thm** Let $\mathcal{H} = (V, E)$ be a hypergraph s.t.

1. $\forall e \in E \Rightarrow |e| \leq n$
2. $\forall e \in E \Rightarrow \exists \{ f \in E : f \cap e = \emptyset, f \neq e \} \leq d$

If $2e(d+1) \leq 2^n$ then $\mathcal{H}$ is 2-colorable.

**pf:**

- Let $V = \{1, 2, 3, 4, 5, \ldots, t\}$
- Let $X_1, X_2, \ldots, X_t \overset{iid}{\sim} B(1, \frac{1}{2})$
- Let $A_e = \{ (\forall v \in e: X_v = 1) \text{ or } (\forall v \in e: X_v = 0) \}$ for $\forall e \in E$

Note that

$\mathbb{P}(A_e) \leq 2^{\mathbb{P}(\forall v \in e: X_v = 1)} \leq 2^{\left(\frac{1}{2}\right)^n} = \frac{1}{e(d+1)}.$

Therefore $\mathbb{P}(\bigcap_{e \in E} \overline{A_e}) > 0.$

QED