Partially Ordered Sets

A poset \((X, P)\): \(X\) is a ground set, \(P\) is a reflexive, antisymmetric, and transitive binary relation on \(X\).

i.e.
1. Reflexive: \((a, a) \in P\) for all \(a \in X\).
2. Antisymmetric: \((a, b), (b, a) \in P \Rightarrow a = b\)
3. Transitive: \((a, b), (b, c) \in P \Rightarrow (a, c) \in P\)

Remark: When \((x, y) \in P\), we write \(x \leq y\) in \(P\).

\(x\) and \(y\) are comparable: if either \(x \leq y\) or \(y \leq x\) in \(P\).

Else they are incomparable.

Notation: \(x < y\) in \(P\) means of \(x \leq y\) in \(P\) and \(x \neq y\).

\(y\) covers \(x\): i.e. 1. \(x < y\) and 2. \(\# z\) s.t. \(x < z < y\) in \(P\).
Basic Terminology (I)

point $x$ is maximal $\equiv \exists y \text{ s.t. } x < y \text{ in } P.$

point $x$ is minimal $\equiv \exists y \text{ s.t. } y < x \text{ in } P.$

Components: A poset is naturally partitioned into components.

Chain: $C \subseteq X$ is called a chain if a pair of points in $C$ is a comparable pair.

Total order $\equiv$ a poset which is a chain.

Maximal chain $\equiv$ no chain $C'$ containing $C$ as a proper subset.

Maximum chain $\equiv$ no chain in $P$ has more points than $C.$

Height of a poset $\equiv$ the $\star$ of points in a maximum chain.

Antichain: $S \subseteq X$ is called an antichain if a pair of points in $S$ is an incomparable pair.

Width of a poset $\equiv$ the $\star$ of points in a maximum antichain.
A poset $P = (X, \leq)$ with $|X| = 21$, height $(X, P) = 6$, width $(X, P) = 8$

$\text{max}(X, P) = \{c, g, d, h, o\}$

$\text{min}(X, P) = \{t, u, s, m, k, l\}$

$X$ can be partitioned into 8 chains:

$C_1 = \{t, r, n, p, o\}$, $C_5 = \{l, i, e, c\}$

$C_2 = \{u, h\}$, $C_6 = \{s\}$

$C_3 = \{m, j, f, b\}$, $C_7 = \{d\}$

$C_4 = \{k, g, a\}$, $C_8 = \{q\}$

$\{a, b, c, d, h, q\}$ is a maximal antichain.

$\{o, p, b, e, i, k\}$ is a maximum chain.
Basic Terminology (II)

\[ \max(x, p) = \text{all maximal points of the poset } (x, p) \]

\[ \min(x, p) = \text{all minimal points of the poset } (x, p) \]

an upper bound for \( S \subseteq X \equiv \text{a } y \in X \text{ s.t. } x \leq y \text{ for all } x \in S. \]

a least upper bound for \( S \), \( \lub(S) = \text{an upper bound } y \text{ for } S \text{ s.t. } y \leq y' \text{ for any other upper bound } y' \text{ for } S. \)

Remark: \( \lub(S) \) may not exist!

Remark: a lower bound for \( S \), a greatest lower bound for \( S \), \( \glb(S) \).

\[ x \vee y = \lub\{x, y\} = \text{join of } x \text{ and } y \]

\[ x \wedge y = \glb\{x, y\} = \text{meet of } x \text{ and } y \]

\[ x \vee Y = \{ x \vee y : x \in X, y \in Y \} \text{ for any } X, Y \subseteq X \]

\[ x \wedge Y = \{ x \wedge y : x \in X, y \in Y \} \text{ for any } X, Y \subseteq X \]
**Lattices**

**Def.** A poset \((X, \mathcal{P})\) is a lattice if \(\text{lub}(S)\) and \(\text{glb}(S)\) exist for any finite non-empty subset \(S\) of \(X\).

The first two posets (poset digrams) are lattices. The last one is not a lattice.

A lattice \((X, \mathcal{P})\) is distributive if for all \(x, y, z \in X\) we have \(x \land (y \lor z) = (x \land y) \lor (x \land z)\).
Basic Terminology (III)

Let \((X, \preceq)\) be a poset and \((L, \leq)\) be a lattice.

- \(x \in L\) is **join-irreducible** if \(x = a \cup b\) yields \(x = a\) or \(x = b\).

\[
J(L) \overset{\text{def}}{=} \text{the set of all nonzero join-irreducible elements of } L,
\]
\(\langle J(L), \leq \rangle\) is regarded as a poset.

- \(S \subseteq X\) is called a **down-set** in \((X, \preceq)\) if \(x \preceq y \in S\) yields \(x \in S\).

- \(\text{V}_I \overset{\text{def}}{=} \text{the join of all elements in } I \subseteq L\) under the partial order of \(L\).

- \(\text{J}(L) \overset{\text{def}}{=} \{ y \in J(L) : y \leq x \}\)

- \(D(J(L)) \overset{\text{def}}{=} \text{the set of all down-set in } \langle J(L), \leq \rangle\).

- Two posets \(X, Y\) are **isomorphic** if \(\exists\) a bijection \(\psi : X \rightarrow Y\) s.t. \(a \preceq b\) in \(X \iff \psi(a) \leq \psi(b)\) in \(Y\).
zero element of the lattice