Section 5.4 (Systems of Linear Differential Equation); Eigenvalues and Eigenvectors

July 1, 2009
A Summary of This Session:
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(1) Finding the eigenvalues and eigenvectors of a $2 \times 2$ matrix.
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(3) Phase-plane method
We are interested in solving systems of first order differential equations of the form:

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or more generally, systems that look like:

\[ x' = f(x, y, t) \]
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Motivation

We are interested in solving systems of first order differential equations of the form:

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\end{align*}
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or more generally, systems that look like:

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In the first case, \(f(x, y)\) and \(g(x, y)\) do not depend on \(t\). They are called autonomous.
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or more generally, systems that look like:

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In the first case, \( f(x, y) \) and \( g(x, y) \) do not depend on \( t \). They are called **autonomous**. In the second case, \( f(x, y, t) \) and \( g(x, y, t) \) depend on \( t \). They are called **non-autonomous**.
Examples

Which of the following examples is autonomous?
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(a): 2 × 2 Systems of Linear Differential Equations
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(a):

\[ x' = 2x - 4xy \]
\[ y' = 2x + 2y^2 \]
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Answer: first-order, autonomous (not linear), 2 × 2 system of dfq’s
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Answer: first-order, autonomous (not linear), 2 \times 2 system of dfq’s

(b)

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x' = 2x + 4y - t \\
y' = x - 2y + \sin t
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(b)

\[ x' = 2x + 4y - t \]
\[ y' = x - 2y + \sin t \]

Answer: first-order, non-autonomous (yet linear), 2 × 2 system of dfq’s

2 × 2 Systems of Linear Differential Equations
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x' = 2x + 4y - t \\
y' = x - 2y + \sin t
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Answer: first-order, non-autonomous (yet linear), 2 \times 2 system of dfq’s

We are interested in **qualitative** as well as **quantitative** descriptions of the solutions.
To find the eigenvalues (and corresponding eigenvectors) of a matrix $A$ means to find the (scalar) values $\lambda$ and corresponding (non-zero) vectors $\vec{v}$ which satisfy the vector equation

$$A\vec{v} = \lambda \vec{v}.$$ 

In some sense the eigenvectors define the main directions along which the matrix $A$ acts (as a geometric transform).
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**Example 1:** Let $A = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix}$. Find its eigenvalues and corresponding eigenvectors.
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**Example 1:** Let $A = \begin{pmatrix} -5 & 2 \\ 1 & -4 \end{pmatrix}$. Find its eigenvalues and corresponding eigenvectors.

We let $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. 
The equation

\[ A \vec{v} = \lambda \vec{v}. \]

means:

\[-5x + 2y = \lambda x \]
\[ x - 4y = \lambda y \]
The equation

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means:

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or

\[ (-5 - \lambda)x + 2y = 0\]
\[x + (-4 - \lambda)y = 0\]
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This system of equations describes the intersection of two lines which go through the origin. In order to have a non-zero solution, the determinant must be zero (this follows from Cramer’s rule). So

$$\begin{vmatrix} (-5 - \lambda) & 2 \\ 1 & (-4 - \lambda) \end{vmatrix} = 0$$
The equation

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\begin{vmatrix}
-5 - \lambda & 2 \\
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\end{vmatrix}
= 0
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Therefore

\[ (-5 - \lambda)(-4 - \lambda) - 2 = 0 \]
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or

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Solving gives: \( \lambda = -3, -6 \).
Therefore
\[(−5 − λ)(−4 − λ) − 2 = 0\]
or
\[λ^2 + 9λ + 20 − 2 = 0\]
That is
\[λ^2 + 9λ + 18 = 0\]
Solving gives: \(λ = −3, −6\).
Now we find the eigenvectors.
For $\lambda_1 = -3$, the system becomes:

$$-2x + 2y = 0$$
$$x - y = 0$$
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Both equations lead to: $x = y$. So we can choose the eigenvector to be $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. 
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For $\lambda_2 = -6$, the system becomes:

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Using eigenvalues and eigenfunctions to solve linear first order systems

This is an alternative method to the annihilator method which explains the nature of the solution obtained.
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Example 2: Solve:

\[ x' = -5x + 2y \]
\[ y' = x - 4y \]
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**Example 2:** Solve:

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Here is how we solve it:

1. Find the matrix \( A \) corresponding to this linear system and put the equation in matrix form \( \mathbf{v}' = A \mathbf{v} \).
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3. The solution vector

\[
\vec{v} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2
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and \( \lambda_2 = -6, \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \).
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Therefore the solution vector is given by:

\[
\vec{v} = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}
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Example 2, cont’d

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This means: \( x(t) = c_1 e^{-3t} - 2c_2 e^{-6t} \) and \( y(t) = c_1 e^{-3t} + c_2 e^{-6t}. \)
Example 2, cont’d

Let’s graph this using **pplane** (http://math.rice.edu/~dfield/dfpp.html). What do you observe?

\[
\begin{align*}
x' &= -5x + 2y \\
y' &= x - 4y
\end{align*}
\]
Example 3

Find the eigenvalues and corresponding eigenvectors of the matrix

\[ A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \]

and use them to write down the solution to

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Make sure to plot the phase plane.
Example 3

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Make sure to plot the phase plane.
The eigenvalues and corresponding eigenvectors are: $\lambda_1 = 7,$

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The eigenvalues and corresponding eigenvectors are: $\lambda_1 = 7$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -1$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. 
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$$\vec{v} = c_1 e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
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This means: \( x(t) = c_1 e^{7t} - c_2 e^{-t} \) and \( y(t) = c_1 e^{7t} + c_2 e^{-t} \).
Example 3, Phase Plane

\[ x' = 3x + 4y \]
\[ y' = 4x + 3y \]