數列 積分

B.5

微積分(I) 期中考試 (2001/12/6)

Problem 1. (4pts)
(a) Prove that $\lim_{n\to\infty}(1+\frac{1}{n^2})^n = 1$.
(b) Find $\lim_{n\to\infty} a_n$, where $a_n = \frac{n!}{n^n}$.

Problem 2. (4pts) Prove that $F(x) = x^2$ is uniformly continuous in $[1,2]$ by finding a function $\delta(\epsilon)$ such that $|F(x)-F(y)| < \epsilon$ whenever $|x-y| < \delta(\epsilon)$.

Problem 3. (4pts) 求同時滿足下面兩條件的二次多項式$f(x) = ax^2+bx+c$.
(a) 對任何一次多項式$g(x) = px+q$, 皆有$\int_{-1}^{1} g(x)f(x)dx = 0$,
(b) $f(1) = 1$.

Problem 4. (10pts) Find the derivative of the following functions:
(a) $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$,
(b) $f(x) = \sqrt[3]{x \div |x|}$.

Problem 5. (4pts) Exprese the following limit as definite integral:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left\{(-1+\frac{3}{n}k)^2+1\right\}\frac{3}{n}$$

Problem 6. (4pts) Determine a polynomial function $F(x)$ satisfying $F'(x)-F(x) = 2x^2$.

Problem 7. (10pts)
(a) Find $D\left\{\int_1^{x^2}\sqrt{1-z^2}dz\right\}$,
(b) Find the following limit (**hint:** 運用(a)的結果與函數微分的定義)

$$\lim_{x\to 1}\frac{1}{x-1}\int_1^{x^2}\sqrt{1-z^2}dz$$

Problem 8. (10pts)
(a) Find $\int_0^1 \sqrt{u}\sqrt{1+u\sqrt{u}}\,du$.
(b) Find $\int_0^n \lfloor x\rfloor\,dx$, where n is a positive integer.

Problem 9. (4pts) If $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational,} \end{cases}$

find

$$\underline{\int_{-1}^{1} f} \quad \text{and} \quad \overline{\int_{-1}^{1} f}.$$

Problem 10. (4pts) Prove that the following limit do not exist.

$$\lim_{x\to 5} x - \lfloor x\rfloor.$$