Exercise Problem Sets 1

Sept. 16. 2022

Problem 1. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, and $a, b \in \mathbb{F}$. Show that $a \leqslant b$ if and only if for all $\varepsilon > 0$, $a < b + \varepsilon$.

Problem 2. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $x, y \in \mathbb{F}$, and $n \in \mathbb{N}$. Show that

1. If $0 \leqslant x < y$, then $x^n < y^n$.

2. If $0 \leqslant x, y$ and $x^n < y^n$, then $x < y$.

Problem 3. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field, $I \subseteq \mathbb{F}$ be an interval, and $f : I \to \mathbb{F}$ be a function.

1. f is said to have a limit at c or we say that the limit of f at c exists if

   (a) there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ with limit c, and

   (b) $\lim\limits_{n\to\infty} f(x_n)$ exists for all convergence sequences $\{x_n\}_{n=1}^{\infty} \subseteq I \backslash \{c\}$ with limit c.

   Show that the limit of f at c exists if and only if there exists $L \in \mathbb{F}$ satisfying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

   $$\big|f(x) - L\big| < \varepsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta \text{ and } x \in I\,.$$

2. f is said to be continuous at a point $c \in I$ if

   $$\lim_{n\to\infty} f(x_n) = f(c) \quad \text{for all convergence sequences } \{x_n\}_{n=1}^{\infty} \subseteq I \text{ with limit } c\,.$$

   Show that f is continuous at c if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

   $$\big|f(x) - f(c)\big| < \varepsilon \qquad \text{whenever} \quad |x - c| < \delta \text{ and } x \in I\,.$$

Problem 4. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an ordered field satisfying the Archimedean property, and $x, y \in \mathbb{F}$ satisfying $x < y$. Show that there exists $r \in \mathbb{Q}$ such that $x < r < y$. This property is called the ***denseness*** of $\mathbb{Q}$ (in Archimedean ordered fields).

Problem 5. Let $(\mathbb{F}, +, \cdot, \leqslant)$ be an Archimedean ordered field, and $0 < \alpha < 1$. Show that $\lim\limits_{n\to\infty} \alpha^n = 0$.