

Exercise Problem Sets 2

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Problem 1. Complete the following.

1. Verify the Wallis's formula: if n is a non-negative integer, then

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x \, dx = \frac{(2^n n!)^2}{(2n+1)!}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}.$$

2. Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$. Show that $\lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = 1$.
3. Let $s_n = \frac{n!}{n^{n+0.5} e^{-n}}$. Show that $\{s_n\}_{n=1}^{\infty}$ is a decreasing sequence.
4. Suppose that you know that $\mathbb{R}$ satisfies the monotone sequence property. Show that $\{s_n\}_{n=1}^{\infty}$ converges and find the limit of $\{s_n\}_{n=1}^{\infty}$.

Hint: 2. Show that $I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ for all $n \in \mathbb{N}$ and then apply the Sandwich lemma.

3. Consider the function $f(x) = \left(1 + \frac{1}{x}\right)^{x+0.5}$.

Problem 2. Let $(\mathbb{F}, +, \cdot, \leq)$ an ordered field satisfying the monotone sequence property, and $y \in \mathbb{F}$ satisfying $y > 1$. Complete the following.

1. Define $y^{1/n}$ properly. (Hint: see how we define $\sqrt[n]{y}$ in class).
2. Show that $y^n - 1 > n(y - 1)$ for all $n \in \mathbb{N} \setminus \{1\}$; thus $y - 1 > n(y^{1/n} - 1)$.
3. Show that if $t > 1$ and $n > (y - 1)/(t - 1)$, then $y^{1/n} < t$.
4. Show that $\lim_{n \rightarrow \infty} y^{1/n} = 1$ as $n \rightarrow \infty$.

Problem 3. Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field satisfying the least upper bound property, and A be a non-empty set of $\mathbb{F}$ which is bounded from below. Define the set $-A$ by $-A \equiv \{-x \in \mathbb{F} \mid x \in A\}$. Prove that

$$\inf A = -\sup(-A).$$

Note that this problem shows that if $\mathbb{F}$ satisfies the least upper bound property, then $\mathbb{F}$ satisfies the greatest lower bound property.

Problem 4. Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field satisfying the least upper bound property, and A be a non-empty set of $\mathbb{F}$ which is bounded from above. Define the set A^{-1} by $A^{-1} \equiv \{x^{-1} \in \mathbb{F} \mid x \in A\}$ whenever $0 \notin A$. Prove that

$$\inf A^{-1} = \frac{1}{\sup A} \quad \text{whenever } A \text{ consists of only positive elements in } \mathbb{F}.$$

Is $\inf A^{-1} = \frac{1}{\sup A}$ in general true?

Problem 5. Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field (satisfying the least upper bound property), and A be a non-empty subset of $\mathbb{F}$. Show that if $a \in A$ is an upper bound for A , then a is the least upper bound for A .

Problem 6. Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field (satisfying the least upper bound property), and A be a non-empty subset of $\mathbb{F}$. Show that if $a \in A$ is an upper bound for A , then a is the least upper bound for A .

Problem 7. Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field satisfying the least upper bound property, and A, B be non-empty subsets of $\mathbb{F}$. Define $A + B = \{x + y \mid x \in A, y \in B\}$. Justify if the following statements are true or false by providing a proof for the true statement and giving a counter-example for the false ones.

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(A + B) = \inf A + \inf B$.
3. $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$.
4. $\sup(A \cap B) = \min\{\sup A, \sup B\}$.
5. $\sup(A \cup B) \geq \max\{\sup A, \sup B\}$.
6. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Problem 8. Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field satisfying the least upper bound property, and $S \subseteq \mathbb{F}$ be non-empty.

1. Show that if S is bounded from below, then

$$\inf S = \sup \{x \in \mathbb{F} \mid x \text{ is a lower bound for } S\}$$

2. Show that if S is bounded from above, then

$$\sup S = \inf \{x \in \mathbb{F} \mid x \text{ is an upper bound for } S\}.$$

Problem 9. Let A, B be two sets, and $f : A \times B \rightarrow \mathbb{F}$ be a function, where $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field satisfying the least upper bound property. Show that

$$\sup_{(x,y) \in A \times B} f(x, y) = \sup_{y \in B} \left(\sup_{x \in A} f(x, y) \right) = \sup_{x \in A} \left(\sup_{y \in B} f(x, y) \right)$$

and

$$\inf_{(x,y) \in A \times B} f(x, y) = \inf_{y \in B} \left(\inf_{x \in A} f(x, y) \right) = \inf_{x \in A} \left(\inf_{y \in B} f(x, y) \right).$$

Problem 10. Let A be a set, and $f, g : A \rightarrow \mathbb{R}$ be two functions. Let $h = \max\{f, g\}$; that is,

$$h(x) = \max\{f(x), g(x)\} \quad \forall x \in A.$$

Show that

$$\sup_{x \in A} h(x) = \max\left\{\sup_{x \in A} f(x), \sup_{x \in A} g(x)\right\}.$$

Generalize the result above to the following: if $f_1, \dots, f_n : A \rightarrow \mathbb{R}$ are real-valued functions, then

$$\sup_{x \in A} \max\{f_1(x), \dots, f_n(x)\} = \max\left\{\sup_{x \in A} f_1(x), \sup_{x \in A} f_2(x), \dots, \sup_{x \in A} f_n(x)\right\}.$$

Can one conclude that if $f_n : A \rightarrow \mathbb{R}$ is a sequence of functions, then

$$\sup_{x \in A} \sup\{f_1(x), \dots, f_n(x), \dots\} = \sup\left\{\sup_{x \in A} f_1(x), \sup_{x \in A} f_2(x), \dots, \sup_{x \in A} f_n(x), \dots\right\}.$$