

## Exercise Problem Sets 3

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**Problem 1.** Let  $a \in \mathbb{R}$ . Define  $a_n$  through the iterated relation

$$a_n = a_{n-1}^2 - a_{n-1} + 1 \quad \forall n > 1, a_1 = a.$$

For what  $a$  is the sequence  $\{a_n\}_{n=1}^\infty$  (1) monotone? (2) bounded?

**Problem 2.** Let  $(\mathbb{F}, +, \cdot, \leq)$  be an ordered field, and  $\{x_n\}_{n=1}^\infty$  be a sequence in  $\mathbb{F}$ . Show that the following three statements are equivalent.

1.  $\{x_n\}_{n=1}^\infty$  converges.
2. Every proper subsequence of  $\{x_n\}_{n=1}^\infty$  converges.
3. Every subsequence of  $\{x_n\}_{n=1}^\infty$  converges.

**Problem 3.** Let  $(\mathbb{F}, +, \cdot, \leq)$  be an Archimedean ordered field,  $\{x_n\}_{n=1}^\infty$  be a sequence in  $\mathbb{F}$ , and  $x \in \mathbb{F}$ . Show the following.

1.  $x$  is a cluster point of  $\{x_n\}_{n=1}^\infty$  if and only if for all  $\varepsilon > 0$  and  $N > 0$  there exists  $n \geq N$  such that  $|x_n - x| < \varepsilon$ .
2.  $x$  is a cluster point of  $\{x_n\}_{n=1}^\infty$  if and only if there exists a subsequence  $\{x_{n_j}\}_{j=1}^\infty$  of  $\{x_n\}_{n=1}^\infty$  converges to  $x$ .

**Remark:** The Archimedean Property is only used in 2 in the direction “ $\Rightarrow$ ”.

**Problem 4.** Let  $(\mathbb{F}, +, \cdot, \leq)$  be an Archimedean ordered field. A number  $x \in \mathbb{F}$  is called an **accumulation point** of a set  $A \subseteq \mathbb{F}$  if for all  $\delta > 0$ ,  $(x - \delta, x + \delta)$  contains at least one point of  $A$  distinct from  $x$ . In logic notation,

$$x \text{ is an accumulation point of } A \iff (\forall \delta > 0)(A \cap (x - \delta, x + \delta) \setminus \{x\} \neq \emptyset).$$

1. Show that if  $\{x_n\}_{n=1}^\infty$  is a sequence in  $\mathbb{F}$  so that  $x_i \neq x_j$  for all  $i, j \in \mathbb{N}$  and  $A = \{x_k \mid k \in \mathbb{N}\}$ , then  $x$  is an accumulation of  $A$  if and only if  $x$  is a cluster point of  $\{x_n\}_{n=1}^\infty$ .
2. How about if the condition  $x_i \neq x_j$  for all  $i, j \in \mathbb{N}$  is removed? Is the statement in 1 still valid?

**Problem 5.** Let  $(\mathbb{F}, +, \cdot, \leq)$  be an Archimedean ordered field, and  $\{x_n\}_{n=1}^\infty \subseteq \mathbb{F}$  be a sequence satisfying  $|x_n - x_{n+1}| < \frac{1}{n}$  for all  $n \in \mathbb{N}$ . Prove or disprove that there exists a subsequence  $\{x_{n_k}\}_{k=1}^\infty$  of  $\{x_n\}_{n=1}^\infty$  so that  $\{x_{n_k}\}_{k=1}^\infty$  is a Cauchy sequence.

**Problem 6.** Let  $(\mathbb{F}, +, \cdot, \leq)$  be an Archimedean ordered field, and  $f : \mathbb{F} \rightarrow \mathbb{F}$  be a function so that

$$|f(x) - f(y)| \leq \alpha |x - y| \quad \forall x, y \in \mathbb{F},$$

where  $\alpha \in \mathbb{F}$  is a constant satisfying  $0 < \alpha < 1$ . Pick an arbitrary  $x_1 \in \mathbb{F}$ , and define  $x_{k+1} = f(x_k)$  for all  $k \in \mathbb{N}$ . Show that  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence in  $\mathbb{F}$ .