

Exercise Problem Sets 4

Oct. 02. 2026

Problem 1. Let $b \in \mathbb{R}$ and $b > 1$.

1. Show the law of exponents holds (for rational exponents); that is, show that

(a) if r, s in $\mathbb{Q}$, then $b^{r+s} = b^r \cdot b^s$.

(b) if r, s in $\mathbb{Q}$, then $b^{r \cdot s} = (b^r)^s$.

2. For $x \in \mathbb{F}$, let $B(x) = \{b^t \in \mathbb{F} \mid t \in \mathbb{Q}, t \leq x\}$. Show that $\sup B(x)$ exists for all $x \in \mathbb{F}$, and $b^r = \sup B(r)$ if $r \in \mathbb{Q}$.

3. Define $b^x = \sup B(x)$ for $x \in \mathbb{F}$. Show that $B(x) > 0$ for all $x \in \mathbb{F}$ and the law of exponents (for exponents in $\mathbb{F}$)

(a) if x, y in $\mathbb{F}$, then $b^{x+y} = b^x \cdot b^y$, (b) if $x, y > 0$, then $b^{x \cdot y} = (b^x)^y$,

are also valid.

4. Show that if $x_1, x_2 \in \mathbb{F}$ and $x_1 < x_2$, then $b^{x_1} < b^{x_2}$. This implies that if x_1, x_2 are two numbers in $\mathbb{F}$ satisfying $b^{x_1} = b^{x_2}$, then $x_1 = x_2$.

5. Let $y > 0$ be given. Show that if $u, v \in \mathbb{F}$ such that $b^u < y$ and $b^v > y$, then $b^{u+1/n} < y$ and $b^{v-1/n} > y$ for sufficiently large n .

6. Let $y > 0$ be given, and $A \subseteq \mathbb{F}$ be the set of all w such that $b^w < y$. Show that $\sup A$ exists and $x = \sup A$ satisfies $b^x = y$. The number x (the uniqueness is guaranteed by 4) satisfying $b^x = y$ is called the logarithm of y to the base b , and is denoted by $\log_b y$.

Hint: Make use of the n -th root defined in class.

Problem 2. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in $\mathbb{R}$. Prove the following inequalities:

$$\begin{aligned} \liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n &\leq \liminf_{n \rightarrow \infty} (x_n + y_n) \leq \liminf_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \\ &\leq \limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n; \\ \left(\liminf_{n \rightarrow \infty} |x_n| \right) \left(\liminf_{n \rightarrow \infty} |y_n| \right) &\leq \liminf_{n \rightarrow \infty} |x_n y_n| \leq \left(\liminf_{n \rightarrow \infty} |x_n| \right) \left(\limsup_{n \rightarrow \infty} |y_n| \right) \\ &\leq \limsup_{n \rightarrow \infty} |x_n y_n| \leq \left(\limsup_{n \rightarrow \infty} |x_n| \right) \left(\limsup_{n \rightarrow \infty} |y_n| \right). \end{aligned}$$

Give examples showing that the equalities are generally not true.

Problem 3. Prove that

$$\liminf_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{|x_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{|x_n|} \leq \limsup_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|}.$$

Give examples to show that the equalities are not true in general. Is it true that $\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|}$ exists implies that $\lim_{n \rightarrow \infty} \frac{|x_{n+1}|}{|x_n|}$ also exists?

Problem 4. Find the following limits.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sqrt[n]{n!}, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sqrt[n]{(n+1)(n+2) \cdots (2n)}.$$

Problem 5. Given the following sets consisting of elements of some sequence of real numbers. Find the limsup and liminf of the sequence.

1. $\{\cos m \mid m = 0, 1, 2, \dots\}$.
2. $\{\sqrt[n]{|\sin m|} \mid m = 1, 2, \dots\}$.
3. $\left\{\left(1 + \frac{1}{m}\right) \sin \frac{m\pi}{6} \mid m = 1, 2, \dots\right\}$.

Hint: 1. First show that for all irrational α , the set

$$S = \{x \in [0, 1] \mid x = k\alpha \pmod{1} \text{ for some } k \in \mathbb{N}\}$$

is dense in $[0, 1]$; that is, for all $y \in [0, 1]$ and $\varepsilon > 0$, there exists $x \in S \cap (y - \varepsilon, y + \varepsilon)$. Then choose $\alpha = \frac{1}{2\pi}$ to conclude that

$$T = \{x \in [0, 2\pi] \mid x = k \pmod{2\pi} \text{ for some } k \in \mathbb{N}\}$$

is dense in $[0, 2\pi]$. To prove that S is dense in $[0, 1]$, you might want to consider the following set

$$S_k = \{x \in [0, 1] \mid x = \ell\alpha \pmod{1} \text{ for some } 1 \leq \ell \leq k+1\}$$

Note that there must be two points in S_k whose distance is less than $\frac{1}{k}$. What happened to (the multiples of) the difference of these two points?

2. Use the fact that π is a Liouville number; that is, there exists $d \in \mathbb{N}$ such that

$$\left|\pi - \frac{p}{q}\right| \geq \frac{1}{q^d} \quad \forall p, q \in \mathbb{Z}, q \neq 0.$$

Problem 6. Let $\|\cdot\| : \mathbb{F}^n \rightarrow \mathbb{R}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, be defined by

$$\|\mathbf{x}\|_p \equiv \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max\{|x_1|, \dots, |x_n|\} & \text{if } p = \infty, \end{cases} \quad \mathbf{x} = (x_1, \dots, x_n).$$

Complete the following.

1. Prove the Hölder inequality $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, where $p, q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.
2. Show that $\|\cdot\|_p$ is indeed a norm on $\mathbb{F}^n$ for all $1 \leq p \leq \infty$.
3. Show that $\|\mathbf{x}\|_\infty = \lim_{p \rightarrow \infty} \|\mathbf{x}\|_p$ for all $\mathbf{x} \in \mathbb{F}^n$.

4. Show that for each $1 \leq p, q \leq \infty$ and $p \neq q$, $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent norms.

Hint: 1. Prove first the Young inequality (if you do not know this inequality)

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \quad \forall a, b \geq 0 \text{ and } p, q \in (1, \infty) \text{ satisfying } \frac{1}{p} + \frac{1}{q} = 1,$$

Problem 7. Complete the following.

1. For $f \in \mathcal{C}([a, b]; \mathbb{R})$, define

$$\|f\|_p = \begin{cases} \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max_{x \in [a, b]} |f(x)| & \text{if } p = \infty. \end{cases}$$

Show that $\|\cdot\|_p$ is a norm on $\mathcal{C}([a, b]; \mathbb{R})$.

2. Show that $\|f\|_\infty = \lim_{p \rightarrow \infty} \|f\|_p$ for all $f \in \mathcal{C}([a, b]; \mathbb{R})$.

3. Are $\|\cdot\|_p$ and $\|\cdot\|_q$ equivalent norms on $\mathcal{C}([a, b]; \mathbb{R})$ for any $1 \leq p, q \leq \infty$?

Problem 8. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Show that for each $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{F}^n$,

$$\|\mathbf{x}\|_1 = \sup_{\|\mathbf{y}\|_\infty = 1} \left| \sum_{k=1}^n x_k y_k \right| = \sup \left\{ \left| \sum_{k=1}^n x_k y_k \right| \mid \|\mathbf{y}\|_\infty = 1 \right\}$$

and

$$\|\mathbf{y}\|_\infty = \sup_{\|\mathbf{x}\|_1 = 1} \left| \sum_{k=1}^n x_k y_k \right| = \sup \left\{ \left| \sum_{k=1}^n x_k y_k \right| \mid \|\mathbf{x}\|_1 = 1 \right\}.$$

Problem 9. Let $\mathcal{V}$ be a vector space (over a scalar field $\mathbb{F}$), and A, B be subset of $\mathcal{V}$. Show that

$$A + B = \bigcup_{\mathbf{a} \in A} (\mathbf{a} + B) = \bigcup_{\mathbf{b} \in B} (\mathbf{b} + A).$$

Problem 10. A set A in a vector space $\mathcal{V}$ is called **convex** if for all $x, y \in A$, the line segment joining x and y , denoted by $\overline{xy}$, lies in A .

1. Show that open r -balls and closed r -balls in any normed vector spaces are convex.
2. Show that an ellipsoid $E \equiv \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T Q \mathbf{x} + \mathbf{b}^T \mathbf{x} + \mathbf{c} \leq 0\}$ is convex, where Q is a positive semi-definite $n \times n$ matrix and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ are vectors.
3. Show that a polytope $P \equiv \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}\}$ is convex, where A is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$ is a vector, and $\leq$ is defined by $\mathbf{c} \leq \mathbf{d}$ if and only if $c_i \leq d_i$ for all components.
4. Show that if C_α is convex for all $\alpha \in I$, then $\bigcap_{\alpha \in I} C_\alpha$ is convex.
5. Show that if $C_1, C_2, \dots, C_N$ are convex sets and $\mu_1, \mu_2, \dots, \mu_N$ are real numbers, then

$$\mu_1 C_1 + \mu_2 C_2 + \dots + \mu_N C_N \equiv \{\mu_1 x_1 + \mu_2 x_2 + \dots + \mu_N x_n \mid x_k \in C_k \text{ for } 1 \leq k \leq N\}$$

is convex.

6. Show that if $C_1, C_2, \dots, C_N$ are convex sets, then the Cartesian product $C_1 \times C_2 \times \dots \times C_N$ is convex.

7. Let A be an $m \times n$ matrix, and C be a convex set in $\mathbb{R}^n$, D be a convex set in $\mathbb{R}^m$. Show that

$$A(C) \equiv \{A\mathbf{x} \mid \mathbf{x} \in C\} \quad \text{and} \quad A^{-1}(D) \equiv \{\mathbf{x} \mid A\mathbf{x} \in D\}$$

are convex.

8. Let $\Delta_k \equiv \left\{ (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \mid 0 \leq \lambda_i \leq 1 \text{ for all } 1 \leq i \leq k \text{ and } \sum_{i=1}^k \lambda_i = 1 \right\}$. A convex combination of k vectors $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n$ is a sum of the form $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_k \mathbf{x}_k$ for some $(\lambda_1, \dots, \lambda_k) \in \Delta_k$. Show that the collection of all convex combinations of k given vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ is convex; that is, show that

$$\left\{ \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \dots + \lambda_k \mathbf{x}_k \mid (\lambda_1, \lambda_2, \dots, \lambda_k) \in \Delta_k \right\}$$

is convex.

9. Let S be a subset in $\mathbb{R}^n$. Show that the collection of all convex combinations of finitely many vectors from S is convex; that is, show that

$$\left\{ \lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k \mid k \in \mathbb{N}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k \in S \text{ and } (\lambda_1, \dots, \lambda_k) \in \Delta_k \right\}$$

is convex. The set defined above is called the convex hull of S and is sometimes denoted by $\text{conv}(S)$.