

Exercise Problem Sets 5

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Problem 1. Show that

$$\|A\|_1 = \max \left\{ \sum_{i=1}^n |a_{i1}|, \sum_{i=1}^n |a_{i2}|, \dots, \sum_{i=1}^n |a_{im}| \right\} \quad \forall A \in \mathcal{M}_{n \times m}.$$

Hint: Mimic the computation of $\|A\|_\infty$ in Example ??, or make use of Exercise Problem ?? and ??.

Problem 2. Let $\mathcal{M}_{n \times m}(\mathbb{F})$ be collection of $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $A \in \mathcal{M}_{n \times m}(\mathbb{F})$, define

$$\|A\|_p = \sup_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}.$$

1. Show that $\|\cdot\|_p$ is a norm on $\mathcal{M}_{n \times m}(\mathbb{F})$.
2. Show that $\|A\|_2 = \sqrt{\text{the maximum eigenvalue of } A^\dagger A}$, where $A^\dagger$ is the conjugate transpose of A .
3. Show that $\|A\|_\infty = \max \left\{ \sum_{k=1}^m |a_{1k}|, \sum_{k=1}^m |a_{2k}|, \dots, \sum_{k=1}^m |a_{nk}| \right\}$ if $A \in \mathcal{M}_{n \times m}(\mathbb{F})$.
4. Show that $\|A\|_1 = \max \left\{ \sum_{k=1}^n |a_{k1}|, \sum_{k=1}^n |a_{k2}|, \dots, \sum_{k=1}^n |a_{km}| \right\}$ if $A \in \mathcal{M}_{n \times m}(\mathbb{F})$.
5. Show that $\|A\|_2^2 \leq \|A\|_1 \|A\|_\infty$ for all $A \in \mathcal{M}_{n \times m}(\mathbb{F})$.

Problem 3. Let $\mathcal{M}_{n \times m}(\mathbb{F})$ be the collection of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Define a function $\|\cdot\|_{p,q} : \mathcal{M}_{n \times m}(\mathbb{F}) \rightarrow \mathbb{R}$ by

$$\|A\|_{p,q} = \sup_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_q,$$

here we recall that $\|\cdot\|_p$ is the p -norm on $\mathbb{F}^n$ given in Problem ??. If $p = q$, we simply use $\|A\|_p$ to denote $\|A\|_{p,q}$. Complete the following.

1. Show that $\|A\|_{p,q} = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_q}{\|\mathbf{x}\|_p}$ for all $p, q \geq 1$.
2. Show that $\|A\|_{p,q} = \inf \{ M \in \mathbb{R} \mid \|A\mathbf{x}\|_q \leq M \|\mathbf{x}\|_p \ \forall \mathbf{x} \in \mathbb{F}^m \}$.
3. $\|A\mathbf{x}\|_q \leq \|A\|_{p,q} \|\mathbf{x}\|_p$ for all $\mathbf{x} \in \mathbb{F}^m$.
4. $\|\cdot\|_{p,q}$ defines a norm on $\mathcal{M}_{n \times m}(\mathbb{F})$.
5. Let $\{A_k\}_{k=1}^\infty \subseteq \mathcal{M}_{n \times m}(\mathbb{F})$. Show that $\lim_{k \rightarrow \infty} \|A_k\|_{p,q} = 0$ if and only if each entry of A_k converges to 0. In other words, by writing $A_k = [a_{ij}^{(k)}]_{1 \leq i \leq n, 1 \leq j \leq m}$, show that $\lim_{k \rightarrow \infty} \|A_k\|_{p,q} = 0$ if and only if $\lim_{k \rightarrow \infty} a_{ij}^{(k)} = 0$ for all $1 \leq i \leq m, 1 \leq j \leq n$. In particular, $A_k \rightarrow A$ in the sense that $\|A_k - A\|_{p,q} \rightarrow 0$ as $k \rightarrow \infty$ if and only if the (i, j) -th entry of A_k converges to (i, j) -th entry of A for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Problem 4. Let $\mathcal{M}_{n \times m}(\mathbb{F})$ be the collection of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Define $\|\cdot\|_F : \mathcal{M}_{n \times m}(\mathbb{F}) \rightarrow \mathbb{R}$ by

$$\|A\|_F \equiv \left(\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

1. Show that $\|A\|_F^2 = \text{tr}(A^\dagger A)$, where $A^\dagger$ is the conjugate transpose of A , and $\text{tr}(M)$ is the trace of square matrix M .
2. Show that $\|\cdot\|_F$ is a norm on $\mathcal{M}_{n \times m}(\mathbb{F})$ (for all $n, m \in \mathbb{N}$). This norm is called the Frobenius norm of matrices.
3. Show that $\|AB\|_F \leq \|A\|_F \|B\|_F$ whenever $A \in \mathcal{M}_{n \times m}(\mathbb{F})$ and $B \in \mathcal{M}_{m \times p}(\mathbb{F})$.

Hint: 3. Let $A = [\mathbf{a}_1 : \mathbf{a}_2 : \dots : \mathbf{a}_m]$ and $B = [\mathbf{b}_1 : \mathbf{b}_2 : \dots : \mathbf{b}_m]^\text{T}$; that is, $\mathbf{a}_k$ is the k -th column of A and $\mathbf{b}_\ell$ is the ℓ -th row of B . Then $AB = \sum_{k=1}^m \mathbf{a}_k \mathbf{b}_k$. First show that $\|\mathbf{a}_k \mathbf{b}_k^\text{T}\|_F = \|\mathbf{a}_k\|_2 \|\mathbf{b}_k\|_2$ and use the triangle inequality to conclude the desired equality.