Exercise Problem Sets 6

Oct. 16. 2026

Problem 1. Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. Show that

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0 \qquad \text{if and only if} \qquad \|\boldsymbol{y}\| \leqslant \|\lambda \boldsymbol{x} + \boldsymbol{y}\| \quad \forall\, \lambda \in \mathbb{F}\,,$$

where $\|\cdot\|$ is the norm induced by the inner product $\langle\cdot,\cdot\rangle$.

Problem 2. Let $(\mathcal{V}, +, \cdot, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{R}$, and define $\|\boldsymbol{v}\| = \langle \boldsymbol{v}, \boldsymbol{v}\rangle^{1/2}$ for all $\boldsymbol{v} \in \mathcal{V}$. Show that

1. $2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2 = \|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2$ (parallelogram law).
2. $\big|\|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2\big| \leqslant \|\boldsymbol{x} + \boldsymbol{y}\|\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$.
3. $4\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2$ (polarization identity).

Can the p-norm $\|\cdot\|_p$ on $\mathbb{R}^n$ be induced from any inner product (on $\mathbb{R}^n$) for $p \neq 2$?

Problem 3. Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{C}$. Show the polarization identity

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \frac{1}{4}\Big(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x} - \boldsymbol{y}\|^2 + i\|\boldsymbol{x} + i\boldsymbol{y}\|^2 - i\|\boldsymbol{x} - i\boldsymbol{y}\|^2\Big) \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}\,.$$

Problem 4. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be three normed vector spaces such that $X, Y \subseteq Z$ and

$$\|\boldsymbol{x}\|_Z \leqslant C\|\boldsymbol{x}\|_X \quad \forall\, \boldsymbol{x} \in X \qquad \text{and} \qquad \|\boldsymbol{y}\|_Z \leqslant C\|\boldsymbol{y}\|_Y \quad \forall\, \boldsymbol{y} \in Y\,.$$

Define

$$E = \big\{\boldsymbol{a} \in Z \,\big|\, \|\boldsymbol{a}\|_E \equiv \max\{\|\boldsymbol{a}\|_X, \|\boldsymbol{a}\|_Y\} < \infty\big\}$$

and

$$F = \big\{\boldsymbol{a} \in Z \,\big|\, \|\boldsymbol{a}\|_F \equiv \inf_{\substack{\boldsymbol{a}=\boldsymbol{x}+\boldsymbol{y}\\ \boldsymbol{x}\in X, \boldsymbol{y}\in Y}} (\|\boldsymbol{x}\|_X + \|\boldsymbol{y}\|_Y) < \infty\big\}\,.$$

1. Show that $(E, \|\cdot\|_E)$ is a normed vector space, and $E = X \cap Y$.
2. Show that $(F, \|\cdot\|_F)$ is a normed vector space. The space F is usually denoted by $X + Y$.

Problem 5. Let (M, d) be a metric space. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}\,.$$

Show that (M, ρ) is also a metric space.

Problem 6. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2\,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2\,, \end{cases} \quad \text{where } \boldsymbol{x} = (x_1, x_2) \text{ and } \boldsymbol{y} = (y_1, y_2).$$

Show that d is a metric on $\mathbb{R}^2$.

Problem 7. Let (M, d) be a metric space, and $\{x_n\}_{n=1}^{\infty}$ be a sequence in M. Show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall\, \varepsilon > 0, \exists\, y \in M \ni \#\big\{n \in \mathbb{N} \,\big|\, x_n \notin B(y, \varepsilon)\big\} < \infty\,.$$

Problem 8. Show that

1. Every convergent sequence in a metric space is a Cauchy sequence.
2. If a subsequence of a Cauchy sequence converges to x, then the sequence converges to x.
3. x is a cluster point of $\{x_k\}_{k=1}^{\infty}$ if and only if $\forall\, \varepsilon > 0$ and $N > 0$, $\exists\, k > N$ with $d(x_k, x) < \varepsilon$.
4. x is a cluster point of $\{x_k\}_{k=1}^{\infty}$ if and only if there is a subsequence converging to x.
5. $x_k \to x$ as $k \to \infty$ if and only if every subsequence of $\{x_k\}_{k=1}^{\infty}$ converges to x.
6. $x_k \to x$ as $k \to \infty$ if and only if every proper subsequence of $\{x_k\}_{k=1}^{\infty}$ has a further subsequence that converges to x.

Problem 9. Let $(\mathcal{V}, \|\cdot\|)$ be a norm space, $\{a_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{V}$, and $\{b_n\}_{n=1}^{\infty}$ be the Cesàro mean of $\{a_k\}_{k=1}^{\infty}$; that is, $b_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k$.

1. Show that if $\{a_k\}_{k=1}^{\infty}$ converges to a, then $\{b_n\}_{n=1}^{\infty}$ converges to a.
2. Is the converse statement "if the Cesàro mean $\{b_n\}_{n=1}^{\infty}$ of a sequence $\{a_k\}_{k=1}^{\infty}$ converges, then the sequence $\{a_k\}_{k=1}^{\infty}$ also converges" true?

Problem 10. Show that $(\mathbb{C}, |\cdot|)$ is complete.

Problem 11. Let A be a subset in $\mathbb{R}^n$. Show that A is bounded if and only if every sequence in A has a convergent subsequence.

Problem 12. Let $\mathcal{V}$ be a vector space over field $\mathbb{F}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be a basis of $\mathcal{V}$; that is, every vector $\mathbf{e} \in \mathcal{V}$ can be expressed (uniquely) by $\boldsymbol{x} = \displaystyle\sum_{i=1}^{n} x_i \mathbf{e}_i$ with $x_i \in \mathbb{F}$ for all $1 \leqslant i \leqslant n$. From Example **??** we know that $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{j=1}^{n} x_i \overline{y_i} \qquad \text{if } \boldsymbol{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j \text{ and } \boldsymbol{y} = \sum_{j=1}^{n} y_j \mathbf{e}_j.$$

is an inner product. Show that $(\mathcal{V}, \|\cdot\|_2)$ is a Banach space, where $\|\cdot\|_2$ is the norm induced by the inner product.