Exercise Problem Sets 7

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Problem 1. Let $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be two sequences of real numbers, and $|x_n - x_{n+1}| < a_n$ for all $n \in \mathbb{N}$. Show that $\{x_n\}_{n=1}^{\infty}$ converges if $\sum_{n=1}^{\infty} a_n$ converges.

Problem 2. Let $\sum_{k=1}^{\infty} a_k$ be a conditionally convergent series. Show that $\sum_{k=1}^{\infty} \big[1 + \operatorname{sgn}(a_k)\big]a_k$ and $\sum_{k=1}^{\infty} \big[1 - \operatorname{sgn}(a_k)\big]a_k$ both diverge. Here the sign function sgn is defined by

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ 0 & \text{if } a = 0\,, \\ -1 & \text{if } a < 0\,. \end{cases}$$

Problem 3. Let $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ ba a sequence. A series $\sum_{k=1}^{\infty} b_k$ is said to be a rearrangement of the series $\sum_{k=1}^{\infty} a_k$ if there exists a rearrangement π of $\mathbb{N}$; that is, $\pi : \mathbb{N} \to \mathbb{N}$ is bijective, such that $b_k = a_{\pi(k)}$.

1. Show that if $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of the series $\sum_{k=1}^{\infty} a_k$ converges and has the value $\sum_{k=1}^{\infty} a_k$.
2. Show that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then for each $r \in \mathbb{R}$, there exists a rearrangement $\sum_{k=1}^{\infty} a_{\pi(k)}$ of the series $\sum_{k=1}^{\infty} a_k$ such that $\sum_{k=1}^{\infty} a_{\pi(k)} = r$.

Problem 4. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n, b_n > 0$ for all $n \geqslant N$. Define

$$c_n = b_n - b_{n+1}\frac{a_{n+1}}{a_n} \qquad \forall\, n \in \mathbb{N}\,. \tag{$\star$}$$

1. Show that if there exists a constant $r > 0$ such that $r < c_n$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ converges.

   Hint: Rewrite $(\star)$ as $b_n = c_n + \frac{a_{n+1}}{a_n} b_{n+1}$ and then obtain

   $$\begin{aligned} b_N &= c_N + \frac{a_{N+1}}{a_N} b_{N+1} = c_N + \frac{a_{N+1}}{a_N}\Big(c_{N+1} + \frac{a_{N+2}}{a_{N+1}} b_{N+2}\Big) = c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} b_{N+2} \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N}\Big(c_{N+2} + \frac{a_{N+3}}{a_{N+2}} b_{N+3}\Big) = \cdots \\ &= c_N + \frac{a_{N+1}}{a_N} c_{N+1} + \frac{a_{N+2}}{a_N} c_{N+2} + \cdots + \frac{a_{N+n}}{a_N} c_{N+n} + \frac{a_{N+n+1}}{a_N} b_{N+n+1}\,. \end{aligned}$$

   Use the fact that $0 < r < c_n$ for all $n \geqslant N$ to conclude that

   $$\sum_{k=N}^{N+n} a_k \leqslant \frac{a_N b_N}{r} \qquad \forall\, n \in \mathbb{N}\,.$$

   Note that then the sequence of partial sum of $\sum_{k=1}^{\infty} a_k$ then is bounded from above $\big($by $\sum_{k=1}^{N-1} a_k + \frac{a_N b_N}{r}\big)$.

2. Show that if $\sum_{k=1}^{\infty} \frac{1}{b_k}$ diverges and $c_n \leqslant 0$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ diverges.

   Hint: The fact that $c_n \leqslant 0$ for all $n \geqslant N$ implies that $b_n a_n \leqslant b_{n+1} a_{n+1}$ for all $n \geqslant N$. Use this fact to conclude that

$$\frac{a_N b_N}{b_n} \leqslant a_n \qquad \forall\, n \geqslant N$$

   and then apply the direct comparison test to conclude that $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 5. Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. We know that the ratio test fails when this happens, but there are some refined results concerning this particular case.

1. **(Raabe's test)**:
   (a) If there exists a constant $\mu > 1$ such that $\frac{a_{n+1}}{a_n} < 1 - \frac{\mu}{n}$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ converges.
   (b) If there exists a constant $0 < \mu < 1$ such that $\frac{a_{n+1}}{a_n} > 1 - \frac{\mu}{n}$ for all $n \geqslant N$, then $\sum_{k=1}^{\infty} a_k$ diverges.

2. **(Gauss's test)**: Suppose that there exist a positive constant $\epsilon > 0$, a constant μ, and a **bounded** sequence $\{R_n\}_{n=1}^{\infty}$ such that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \qquad \text{for all } n \geqslant N\,.$$

   (a) If $\mu > 1$, then $\sum_{k=1}^{\infty} a_k$ converges. (b) If $\mu \leqslant 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 6. Let a_n be defined by $a_n = \begin{cases} \frac{n+1}{2^n} & \text{if } n \text{ is odd}\,, \\ \frac{n}{3^n} & \text{if } n \text{ is even}\,. \end{cases}$ Compute the value of $\liminf_{n\to\infty} \sqrt[n]{a_n}$, $\limsup_{n\to\infty} \sqrt[n]{a_n}$, $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$, and conclude that whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or not.

Hint: You can use $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \sqrt[n]{n+1} = 1$ without proving it.

Problem 7. Let $\alpha \in \mathbb{R}$, $\alpha > \frac{1}{3}$. Discuss the absolute convergence or the conditional convergence of the series $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^\alpha + (-1)^k}$.

Problem 8. Determine whether the following series converge or not. Also test for their absolute convergence.

1. $\sum_{n=1}^{\infty} \sin(n^{-\alpha})$, $\alpha > 0$;

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\log(n+1)-\log n}{\arctan\frac{2}{n}}$;

3. $\displaystyle\sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{1\cdot 2\cdots n\cdot c(c+1)\cdots(c+n-1)}$;

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}\big(1+\frac{1}{3}+\cdots+\frac{1}{2n+1}\big)$;

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$;

The a, b, c in (3) are real number except negative integers.

Problem 9. Complete the following.

1. Show that $\displaystyle\sum_{k=1}^{\infty}\Big(1-\frac{1}{\sqrt{k}}\Big)^k$ converges.
2. Show that $\displaystyle\sum_{k=2}^{\infty}\frac{\log(k+1)-\log k}{(\log k)^2}$ converges.
3. Use Gauss's test to show that both the general harmonic series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{ak+b}$, where $a\neq 0$, and the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ diverge.
4. Show that $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$ converges if $\alpha>1$ and diverges if $\alpha\leqslant 1$.
5. Test the following "hypergeometric" series for convergence or divergence:
   (a) $\displaystyle\sum_{k=1}^{\infty}\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k-1)}=\frac{\alpha}{\beta}+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}+\cdots$.
   (b) $\displaystyle 1+\frac{\alpha\cdot\beta}{1\cdot\gamma}+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\gamma\cdot(\gamma+1)}+\frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}+\cdots$.

Problem 10. Consider the function $f(x)=\displaystyle\sum_{k=1}^{\infty}\frac{\sin(kx)}{k}$.

1. Find the domain of f.
2. Show that for each $\varepsilon>0$ and $0<\delta<\pi$, there exists $N>0$ and N depends only on ε and δ but is independent of x, such that

$$\Big|\sum_{k=n}^{n+p}\frac{\sin(kx)}{k}\Big|<\varepsilon \qquad \forall\, n\geqslant N, p\geqslant 0 \text{ and } x\in[\delta, 2\pi-\delta]\,.$$